MULTIPLE PUBLIC GOODS IN NETWORKS

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Abstract. In this paper we consider an n-player simultaneous move game on a fixed network, in which each player chooses her investment level in each of m goods that are non-rivalrous and non-excludable across links in the network. We analyze the existence, stability and welfare properties of PSNEs of the game. Our results demonstrate that while every game necessarily has a specialized equilibrium, the stability of equilibrium profiles and the existence of specialized equilibria in which specialization is dispersed depend crucially on the network structure. We also provide some interesting welfare implications relating to concentration of specialization.

1. Introduction

In this paper we consider an n-player simultaneous move game on a fixed network, in which each player or agent chooses her investment level in each of m goods that are non-rivalrous and non-excludable across links in the network. Our analysis is an extension of the single good model proposed by Bramoullé and Kranton (2007) to include multiple goods. Their model assumes a fixed network and agents choose levels of investment on a single good that is non-rivalrous and non-excludable across links in the network. The primary working example adopted by them is innovation, the results of which are non-excludable in certain dimensions. Under the assumption of diminishing marginal utility, they identify three kinds of Nash equilibrium profiles - (i) *distributed equilibria*, where all agents in the network make positive investments, (ii) *specialized equilibria*, where some agents in the network make zero investment, free-riding on the investments made by *specialists*, and (iii) *hybrid equilibria*, that fall between these two extremes. Bramoullé and Kranton find that *specialized equilibria are the only stable outcomes*, and the existence of specialized equilibria is characterized (and ensured) by the existence of maximal independent sets of graphs (that represent networks). More specifically, they find that a specialized strategy profile is a Nash equilibrium if and only if its set of specialists is a maximal independent set of the given network. In graph theory, an independent set of a graph is a set of agents such that no two agents who belong to the set are linked to each other. A *maximal* independent set is an independent set that is not a strict subset of another independent set, and every graph has at least one maximal independent set. Thus they find that there always exists a specialized Nash equilibrium because for every graph, there always exists a maximal independent set.

Our model is suited to contexts where entities invest in various kinds of goods (for example, information or knowledge creation) that are non-excludable in certain dimensions, each with its own associated marginal cost of investment. In this multiple-goods framework, we retain the assumption of strategic substitutability between investment made by an agent, on any particular good, and those made in the same good by her neighbours. Additionally, we assume that there is no interaction between the various goods in agents' utility function. That is, the utility function for any agent is additively separable in the various goods. Because of this simple extension, most of our results are also extensions to Bramoullé and Kranton's results. We analyse pure strategy Nash equilibria and find that equilibria that are specialized in any subset of goods are characterized by all the sets of specialists being maximal independent sets of the graph

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(representing the network) in question. We also find that only specialized equilibria may be stable.

The motivation behind our extension as well as our primary contribution lies in the analysis of specialized equilibria with varying degrees of concentration of specialization. In our model different investment profiles may have differing degrees of concentration of specialization, with profiles where some agents specialize in all goods and everyone else free-rides being one extreme and profiles in which everyone specializes in an equal number of goods being the other. We are interested in investigating the existence of Nash equilibria with *limited* degrees of concentration of specialization. The relevance of this line of investigation comes from the literature on *reciprocity* and *fairness*, expectations thereof, and the role that these play in various public goods contexts. There is, in fact, a significant body of literature that emphasizes the importance of norms and expectations of fairness and reciprocity in people's decision making in contexts of public goods and other social dilemmas. Marwell and Ames (1979) and Dawes (1980) suggested that decision making in social dilemmas is substantially influenced by considerations of fairness. Van Dijk and Wilke (1984) studied the effect of conditional contributions on the provision of public goods by examining an experiment in which one member (of a five person group) supposedly committed him/herself to contribute only if at least one other member contributed too. They found that group members reciprocated the motivation that they ascribed to this conditional contributor. Eek, Beil, and Grling (1999) established, through an examination of evidence based on experiments, that norms about distributive justice (especially those of equity and equality) play an important role in determining people's behaviour when deciding whether or not to contribute to a common good. Kurzban and DeScioli (2008) have identified, within experiments on public goods games, *reciprocal players* who seek to access information about median contribution levels before deciding on how much to contribute themselves, and were even willing to pay to acquire this information. It is apparent that these players value reciprocity very highly within public goods context. Further, there exists a rich body of experimental literature on public good contributions that shows that peer-punishment can sustain cooperation in public goods game, and players punish free-riders by either withdrawing/reducing their own public good contributions (in repeated interactions) or explicitly adopting punishment strategies where available, even if they are costly[1]. This body of literature suggests the improbability of outcomes where investment in a large number public goods is concentrated in the hands of a few specialists. Especially in network contexts, where links may be broken (people may *refuse* to share information/knowledge creation without some degree of reciprocity) by the entities involved in the network, these robust findings on the influence of fairness, reciprocity and equity norms on public good contributions necessitate the search for equilibria with lower/limited degrees of concentration of specialization, as opposed to highly concentrated specialized equilibria where a small number of people specialize in all goods. There are two ways in which we limit the degree of concentration of specialization in specialized Nash equilibria. The first is by restricting the number of goods that any agent can specialize in, and the second is by ensuring that no agent is a complete free-rider, such that every agent specializes in some good. We then attempt to characterize these specialized equilibria and find that their existence depends on the existence of sets of maximal independents sets of the network with certain desirable properties.

In the context of multiple public goods, There exists a body of literature that explores questions around voluntary contributions and efficiency in production/provision in the context of multiple public goods. Mutuswami and Winter (2004) have explored sequential mechanisms for efficient production of multiple public goods, while Cherry and Dickinson (2007) have focused on the patterns of individual contributions in presence of multiple, *competing* public goods. Ghosh et al

[1]Ostrom et al (1992) allowed for costly punishment in a repeated common pool resource game to find that this did indeed motivate appropriators to develop credible commitments to cooperate. Fehr and Gchter (2000) have demonstrated experimentally the widespread willingness of cooperators/public good contributors to punish free-riders. Maier-Rigaud, Martinsson, and Staffiero (2010) have analysed the positive effects of ostracism on cooperation (increased contribution levels) in a linear public good experiment.

(2007) have extended the model of voluntary contributions to multiple public goods by allowing or bundling of the goods such that agents contribute to a common pool, which is then allocated towards the financing of two pure public goods. Richefort (2018) has explored a voluntary contribution game with multiple public goods (each benefiting a different group of players) in presence of warm-glow effects of giving, while Chan and Wolk (2020) have studied the effect of choice environment on contribution behaviour in settings with multiple public goods. Our paper also contributes to this body of literature on multiple public goods.

The remaining paper is organized in the following manner. Our model is formally presented in the next section, followed by equilibrium analysis in the third section. The fourth section presents welfare analysis, and the final section concludes the paper. All proofs are presented within relevant sections with appropriate discussion and examples.

2. Model

$N = \{1, 2, \ldots, n\}$ is the set of agents in a fixed network represented as a graph g, where $g_{ij} = 1$ if agent i is linked to agent j and $g_{ij} = 0$ otherwise. $M = \{1, 2, \ldots, m\}$ is the set of non-rivalrous and non-excludable goods being shared across social links. Since benefits of public goods flow both ways, the fixed network is considered to be undirected and we have $g_{ij} = g_{ji}$. Thus, for any pair of agents i and j, $g_{ij} = g_{ji} = 1$ implies that both i and j have access to each others' investments in all goods. The *neighbourhood* of any agent i in network g is the set $N_i(g) = \{j \in N \mid g_{ij} = g_{ji} = 1\}$.

$x_{ip} \geq 0$ denotes agent i's choice of investment in good p, and $c_p > 0$ denotes the marginal cost of investment in good p.
$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im}) \in \Re^m_+$ is the vector of i's investments in each good in M.
$\mathbf{x}^p = (x_{1p}, x_{2p}, \ldots, x_{np}) \in \Re^n_+$ is the vector of each agent's investment in good p.
$\mathbf{x} = (x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2m}, \ldots, x_{n1}, \ldots, x_{nm})$ is a complete strategy profile.
$\overline{x}_{ip} = \sum_{j \in N_i(g)} x_{jp}$ denotes agent i's access to good p through i's neighbours in the network.
The utility derived by agent i from strategy profile $\mathbf{x}$ over graph g is given by the function

$$U_i(\mathbf{x}, g) = F(x_{i1} + \overline{x}_{i1}, x_{i2} + \overline{x}_{i2}, \ldots x_{im} + \overline{x}_{im}) - \sum_{p \in M} c_p x_{ip}$$

where for any $p \in M$, the first order partial derivative $F_p() > 0$ and the second order partial derivative $F_{pp}() < 0$. Further, for any $p \neq r$, the cross-partial derivative $F_{pr}() = 0$. That is, the utility function is increasing in all arguments, and the conditions on second order partial and cross partial derivatives ensure that it is also concave. This implies there is perfect substitutability between one's own investment and one's neighbours' investments in any good, and no interaction between goods in one's utility.

Note that the assumption of diminishing marginal benefits and fixed marginal costs for all goods ensures the existence of the vector $x^* = (x^*_1, x^*_2, \ldots, x^*_m) \in \Re^m_+$ such that $x^*_p = arg[F_p(\cdot) = c_p]$ for any $p \in M$. x^* is a vector of *optimal* amounts of each good that any agent would, *in isolation*, choose to invest in. This is the same for all agents under the assumption that the utility function and marginal cost for any good is the same for all agents.

Consider any profile $\mathbf{x}$.

Definition 1. *$i \in N$ is a p-specialist iff $x_{ip} = x^*_p$.*

Let $S_p(\mathbf{x}) = \{i \in N \mid x_{ip} = x^*_p\}$. This is the set of all *specialists* in good p. For all $i \in N$ let $Q_i(\mathbf{x}) = \{p \in M \mid x_{ip} = x^*_p\}$ and suppose $|Q_i(\mathbf{x})| = q_i$. Here, $Q_i(\mathbf{x})$ is the set of all goods that agent i specializes in.

Definition 2. *Profile $\mathbf{x}$ is p-specialized iff $(\forall i \in N)(x_{ip} \in \{0, x^*_p\})$*

Definition 3. *Profile* $\mathbf{x}$ *is p-distributed iff* $(\forall i \in N)(0 < x_{ip} < x_p^*)$

Thus profiles specialized in a certain good are those profiles where agents either invest in optimal amounts of the good or free-ride by making zero investment in the good. On the other hand, distributed profiles have all agents investing positive amounts in the good(s) that it is distributed in, without anyone investing an optimal amount. Further, a strategy profile could be specialized in some goods and distributed in others.

Definition 4. *Profile* $\mathbf{x}$ *is Q-specialized iff* $(\forall i \in N)(\forall p \in Q)[x_{ip} \in \{0, x_p^*\}]$.

Let $Q(\mathbf{x}) = \{p \in M \mid (\forall i \in N)[x_{ip} \in \{0, x_p^*\}]\}$ and let $q = |Q|$. Here $Q(\mathbf{x})$ is the set of all goods for which profile $\mathbf{x}$ is specialized.

Definition 5. $J \subseteq N$ *is an independent set of graph* g *iff* $(\forall i, j \in J)(g_{ij} = g_{ji} = 0)$. *An independent set is maximal when it is not a proper subset of any other independent set.*

Independent set of a graph, a widely used concept in graph theory, refers to a set of nodes which are all mutually unlinked in the graph. A useful property of *maximal* independent sets is that every agent either belongs to the set or is linked to an agent in the set. Further, a *maximal independent set of order* r is a maximal independent set such that every agent who does not belong to the set is linked to *atleast* r agents in the set.

Definition 6. *For any graph* g*,* $I(g)$ *is the set of all maximal independent sets of* g*. A collection of maximal independent sets of* g *is a subset of* $I(g)$*.*

3. Equilibrium analysis

We begin by analysing the access to public goods that agents must have in Nash equilibrium. Given perfect substitutability between agents' investment and the investment of neighbours, along with concavity of the utility function, we have the following lemma.

Lemma 1 If $\mathbf{x}$ is a Nash equilibrium profile, then for any agent i and good p it is necessary that $x_{ip} + \bar{x}_{ip} \geq x_p^*$. Further, if $x_{ip} > 0$, then $x_{ip} + \bar{x}_{ip} = x_p^*$

Proof: Suppose $\mathbf{x}$ is a Nash equilibrium profile in which there is an agent i and a good p such that $x_{ip} + \bar{x}_{ip} < x_p^*$. This implies that $F_p(x_{ip} + \bar{x}_{ip}) > c_p$ and agent i can increase her utility by increasing her own investment in good p. In fact, i can optimally increase her investment to a value x'_{ip} such that $F_p(x'_{ip} + \bar{x}_{ip}) = c_p$. This contradicts the fact that $\mathbf{x}$ is a Nash equilibrium.

Next, suppose there is an agent i and good p such that $x_{ip} > 0$ and $x_{ip} + \bar{x}_{ip} > x_p^*$. This implies that $F_p(x_{ip} + \bar{x}_{ip}) < c_p$ and agent i can strictly increase her utility by reducing her investment in good p. This contradicts the fact that $\mathbf{x}$ is a Nash equilibrium. Lemma 1 is proved by contradiction. □

Thus, in any Nash equilibrium profile every agent must have access to *at least* the optimal amount x_p^* of any good p and if an agent has access to *more* than this amount, then she must not herself be investing in the good.

Proposition 1 If $\mathbf{x}$ is a Q-specialized Nash equilibrium then $S_p(\mathbf{x})$ is a maximal independent set for all $p \in Q$.

Proof: Let $\mathbf{x}$ be Q-specialized profile. Suppose that for good $p \in Q$, $S_p(\mathbf{x})$ is not a maximal independent set.This implies $(\exists i \in S_p(\mathbf{x}))[N_i(g) \cap S_p(\mathbf{x}) \neq \phi]$ or $(\exists i \in N - S_p(\mathbf{x}))[N_i(g) \cap S_p(\mathbf{x}) = \phi]$. Suppose $(\exists i \in S_p(\mathbf{x}))[N_i(g) \cap S_p(\mathbf{x}) \neq \phi]$, *i.e.* $S_p(\mathbf{x})$ is not an independent set and there exists atleast one neighbour of i who also belongs in $S_p(\mathbf{x})$. This means that $x_{ip} + \bar{x}_{ip} \geq 2x_p^*$ and contradicts lemma 1.

Suppose $(\exists i \in N - S_p(\mathbf{x}))[N_i(g) \cap S_p(\mathbf{x}) = \phi]$, *i.e.* $S_p(\mathbf{x})$ is not a *maximal* independent set and there exists an agent i who is not in $S_p(\mathbf{x})$ and is also not linked to any agent in $S_p(\mathbf{x})$. By lemma 1, this must imply $x_{ip} = x_p^*$. But this contradicts that $i \notin S_p(\mathbf{x})$. Therefore if $S_p(\mathbf{x})$ is not a maximal independent set then $\mathbf{x}$ is not a Nash equilibrium. Proposition 1 is proved by contradiction. □

In an equilibrium profile that is specialized in any good p the set of specialists must not be linked to each other, else they have incentive to reduce personal investment, as discussed in lemma 1. Thus the set of specialists must form an independent set. Further, all agents who free-ride in good p must be linked to atleast one specialist (to gain access to good p) and hence the set of specialists must be maximally independent. This is driven by the perfect substitutability between agents' own investment and investment made by neighbours.

Proposition 2 If $\mathbf{x}$ is an M-specialized profile such that for any $p \in M$, $S_p(\mathbf{x})$ is a maximal independent set, then it implies that $\mathbf{x}$ is an M-specialized Nash equilibrium.

Proof: Let $\mathbf{x}$ be M-specialized profile such that for every $p \in M$, $S_p(\mathbf{x})$ is a maximal independent set. Consider an arbitrary $i \in N$ and $p \in M$. If $i \in S_p(\mathbf{x})$, then because $S_p(\mathbf{x})$ is a maximal independent set, it must be that $(\forall j \in N_i(g))[j \notin S_p(\mathbf{x})$ and $x_{jp} = 0]$. Thus i has no incentive to change her investment in good p. On the other hand, if $i \notin S_p(\mathbf{x})$ (that is, $x_{ip} = 0$) then since $S_p(\mathbf{x})$ is a maximal independent set, i must be linked to atleast one person in $S_p(\mathbf{x})$, *i.e.* $x_{ip} + \bar{x}_{jp} \geq x_p^*$. Thus i has no incentive to increase her investment in good p. Since any arbitrary agent has no incentive to unilaterally deviate, the strategy profile $\mathbf{x}$ is a Nash equilibrium. □

Proposition 3 For any graph g there always exists a M-specialized Nash equilibrium.

Proof This follows immediately for Proposition 1 and the fact that for every graph there exists a maximal independent set.

Our analysis is crucially driven by the fact that for every good, we can identify clearly an *optimal* level of investment such that if accessing less than this level, agents find that some further investment in the good more than justifies the additional cost and increases utility and if accessing more than this level agents can increase net utility by reducing their own investment and cost incurred. Note that given agents' locations in the fixed network and the investment levels for various goods made by one's neighbours, an agent can make *independent* decisions about her optimal investment in different goods. Since there is no interaction between goods in the utility function, every agent's *optimal investment in one good is independent of the optimal investment in any other good*; rather it is influenced only by their neighbours' investment in that good and hence also by their location in the network. Thus we can separate out agents' best responses for different goods in the following way.

Best responses Let $b_{ip}(\mathbf{x})$ denote agent i's best response or optimal investment with respect to good p for strategy profile $\mathbf{x}$. Then, from lemma 1 we can deduce that

$$(\forall i \in N)(\forall p \in M)(b_{ip}(\mathbf{x}) = max\{x_p^* - \bar{x}_{ip}, 0\})$$

That is, if the access for good p that agent i gets from her neighbours exceeds the optimal level x_p^*, then i's best response is to make no investment in the good. If however, access from neighbours falls short of the optimal amount, i's best response is to invest in the difference, compensating for the shortfall. We introduce some notation to discuss best response dynamics.

Given a strategy profile $\mathbf{x}$, let $\mathbf{b^p}(\mathbf{x}) = (b_{1p}(\mathbf{x}), b_{2p}(\mathbf{x}) \dots b_{np}(\mathbf{x}))$ be the vector of all agents' best responses for good p and $\mathbf{b_i}(\mathbf{x}) = (b_{i1}(\mathbf{x}), b_{i2}(\mathbf{x}) \dots b_{im}(\mathbf{x}))$ be the vector of agent i's best responses for all goods.

Let $\mathbf{B}(\mathbf{x}) = (b_{11}(\mathbf{x}), \dots b_{1m}(\mathbf{x}), b_{21}(\mathbf{x}), \dots b_{2m}(\mathbf{x}), \dots b_{n1}(\mathbf{x}), \dots b_{nm}(\mathbf{x}))$ denote all agents' best responses for all goods, in response to strategy profile $\mathbf{x}$.

Let $\mathbf{B} \circ \mathbf{B} \circ \dots \mathbf{B} \circ \mathbf{B}(\mathbf{x})$ with the composition taken k times be denoted as $B^k(\mathbf{x})$.

Lemma 2 If $\mathbf{x} \leq \mathbf{x}'$, then $\mathbf{B} \circ \mathbf{B}(\mathbf{x}) \leq \mathbf{B} \circ \mathbf{B}(\mathbf{x}')$

Proof Suppose given two strategy profiles $\mathbf{x}$ and $\mathbf{x}'$, for some good $p \in M$ we have $x_{ip} \leq x'_{ip}$ for all $i \in N$. Then, for any arbitrary agent i, it follows that $x_p^* - \bar{x}_{ip} \geq x_p^* - \bar{x}'_{ip}$. This implies that $max\{x_p^* - \bar{x}_{ip}, 0\} \geq max\{x_p^* - \bar{x}'_{ip}, 0\}$, or, $\mathbf{b^p}(\mathbf{x}) \geq \mathbf{b^p}(\mathbf{x}')$. Applying $\mathbf{b^p}$ again to this inequality gives us:

If $\mathbf{x_p} \leq \mathbf{x'_p}$, then $\mathbf{b^p} \circ \mathbf{b^p}(\mathbf{x_p}) \leq \mathbf{b^p} \circ \mathbf{b^p}(\mathbf{x'_p})$.

Further, since every agent's optimal choice (or best response) of investment in every good is independent of their optimal choice of investment in all other goods, the above analysis should hold simultaneously for *all goods* in M. This proves that if $\mathbf{x} \leq \mathbf{x'}$, then $\mathbf{B} \circ \mathbf{B}(\mathbf{x}) \leq \mathbf{B} \circ \mathbf{B}(\mathbf{x'})$. □

With this understanding of best response functions and the characterization of specialized equilibrium profiles, we turn our attention to stability of Nash equilibria. Following Bramoullé and Kranton, our idea of stability is based on Nash tâtonnement. The stability of an equilibrium profile is defined by the ability of best response dynamics to cause a convergence back to the profile in case of small perturbations.

Consider a perturbation $\epsilon \in (\Re^m_+)^n$ on an equilibrium profile $\mathbf{x}$ which perturbs any arbitrary agent i's investment in good p by ϵ_{ip}. The equilibrium profile $\mathbf{x}$ is stable if and only if there exists for any good p, a value δ_p such that as long as every perturbation ϵ_{ip} is capped in magnitude by δ_p, the best response dynamics after said perturbation will follow a trajectory such that we can identify an integer K such that beyond Kth time period, the best responses of all agents are identical to the original strategy profile $\mathbf{x}$. The formal definition is as follows.

Definition 7 *Equilibrium profile* $\mathbf{x}$ *is said to be stable iff*

$$\left(\exists \delta \in \Re^m_{++}\right)\left(\forall \epsilon \in (\Re^m_+)^n\right)\Big[(\forall i \in N)(\forall p \in M)(x_{ip} + \epsilon_{ip} \geq 0 \wedge |\epsilon_{ip}| \leq \delta_p) \longrightarrow (\exists K)(\forall k \geq K)(b^k(\mathbf{x} + \epsilon) = \mathbf{x})\Big]$$

Proposition 4 $\mathbf{x}$ is a stable equilibrium iff $\mathbf{x}$ is M-specialized and $S_p(\mathbf{x})$ is a maximal independent set of order 2 for all $p \in M$.

The argument of the proof is as follows. First it is argued that non specialized equilibria (distributed as well as hybrid) are not stable. If an equilibrium strategy profile is not specialized in atleast one good p, and investments for individuals acquiring positive (but less than optimal) amounts of p are increased very slightly, then *all* of their neighbours respond by reducing their own investments in p, which causes the former agents to further increase their investment; this chain of best responses continues to create an increasing divergence from the original strategy profile.

Next, we examine specialized equilibria where the set of specialists in good p is maximally independent of order 1. This means that there exists an agent j in $N - S_p(\mathbf{x})$ who is linked to only one specialist, say k, in $S_p(\mathbf{x})$. We consider a perturbation where j's investment in p is increased very slightly. Then, agent k responds by reducing her investment by that amount. This causes j and all other agents who are linked to *only* k in $S_p(\mathbf{x})$ to increase their investments, to which k again responds by further reducing her investment. The chain of best responses continues to create a divergence from the original strategy profile and such equilibria are not stable.

Finally, we note that if all free-riders were linked to atleast *two* specialists in $S_p(\mathbf{x})$, then a small decrease in any specialist's investment does not translate to further increases in their neighbours' investments because being connected to two specialists ensures that for small enough perturbations, every free-rider still has access to the optimal amount of the concerned good. Thus specialized equilibria where every free-rider is linked to atleast two specialists are stable. The formal proof is given below.

Proof of Proposition 4: Consider an equilibrium profile $\mathbf{x}$ which is not M-specialized. Choose $q \in M$ for which there exists $i \in N$ such that $0 < x_{iq} < x^*_q$. Let $J = \{i \in N \mid 0 < x_{iq} < x^*_q\}$. Choose $\epsilon \in (\Re^m_+)^n$ such that $(\forall i \in J)(\epsilon_{iq} = \delta_q \geq 0) \wedge (\forall i \in N - J)(\epsilon_{iq} = 0) \wedge (\forall p \in M - \{q\})(\forall i \in N)(\epsilon_{ip} = 0)$.

For all $i \in J$, $b_{iq}(\mathbf{x} + \epsilon) = x_{iq} - \bar{\epsilon}_{iq}$ and $b^2_{iq}(\mathbf{x} + \epsilon) = x_{iq} + \sum\limits_{l \in N_i(g) \cap J} \bar{\epsilon}_{lq} \geq x_{iq} + \epsilon_{iq}$

For all $i \in N - J$, $x_{iq} = 0 \rightarrow b_{iq}(\mathbf{x}+\epsilon) = 0$ and $b^2_{iq}(\mathbf{x}+\epsilon) \geq 0 = x_{iq} + \epsilon_{iq}$

For all $i \in N - J$, $x_{iq} = x^*_q \rightarrow b_{iq}(\mathbf{x}+\epsilon) = x^*_q$ and $b^2_{iq}(\mathbf{x}+\epsilon) = x^*_q \geq x_{iq} + \epsilon_{iq}$

For all other goods $p \in M - \{q\}$ and $i \in N$, $b_{ip}(\mathbf{x}+\epsilon) = x_{ip}$ and $b^2_{ip}(\mathbf{x}+\epsilon) = x_{ip} = x_{ip} + \epsilon_{ip}$

Thus $\mathbf{B}^2(\mathbf{x}+\epsilon) \geq \mathbf{x}+\epsilon$. This in view of lemma 2 implies that $\mathbf{B}^{2k}(\mathbf{x}+\epsilon) \geq \mathbf{x}+\epsilon$. Therefore $\mathbf{x}$ is not stable.

Consider an equilibrium profile $\mathbf{x}$ which is M-specialized but $S_p(\mathbf{x})$ is not a maximal independent set of order 2 for all $p \in M$. Choose $q \in M$ and $j \in N$ such that $j \notin S_q(\mathbf{x})$ and $N_j(g) \cap S_q(\mathbf{x}) = \{k\}$. That is, agent j is linked to only a single agent k in $S_q(\mathbf{x})$. Choose $\epsilon \in (\Re^m_+)^n$ such that $\epsilon_{jq} = \delta_q > 0 \wedge (\forall i \in N - \{j\})(\epsilon_{iq} = 0) \wedge (\forall p \in M - \{q\})(\forall i \in N)(\epsilon_{ip} = 0)$.

$b_{jq}(\mathbf{x}+\epsilon) = 0 = x_{jq}$, $b_{kq}(\mathbf{x}+\epsilon) = x_{kq} - \delta_q$ and for all $i \in N - \{j,k\}$, $b_{iq}(\mathbf{x}+\epsilon) = x_{iq}$.

$b^2_{kq}(\mathbf{x}+\epsilon) = x_{kq}$, $(\forall i \in \{l \in N - \{k\} \mid k \in N_l(g) \cap S_q(\mathbf{x})\})[b^2_{ip}(\mathbf{x}+\epsilon) = x_{iq} + \delta_q]$ and $(\forall i \in N - \{l \in N - \{k\} \mid k \in N_l(g) \cap S_q(\mathbf{x})\} \cup \{k\})[b^2_{ip}(\mathbf{x}+\epsilon) = x_{iq}]$.

For all $i \in N$, $b_{ip}(\mathbf{x}+\epsilon) = x_{ip}$ and $b^2_{ip}(\mathbf{x}+\epsilon) = x_{ip} = x_{ip} + \epsilon_{ip}$.

Thus $\mathbf{B}^2(\mathbf{x}+\epsilon) \geq \mathbf{x}+\epsilon$. This in view of lemma 2 implies that $\mathbf{B}^{2k}(\mathbf{x}+\epsilon) \geq \mathbf{x}+\epsilon$. Therefore $\mathbf{x}$ is not stable.

Now suppose $\mathbf{x}$ is M-specialized and $S_p(\mathbf{x})$ is a maximal independent set of order 2 for all $p \in M$. Let $\delta_p = \dfrac{x^*_p}{n^2}$. Take any $\epsilon \in (\Re^m_+)^n$ such that $(\forall i \in N)(\forall p \in M)(|\epsilon_{ip}| \leq \delta_p \wedge x_{ip} + \epsilon_{ip} \geq 0)$.

Let $\mathbf{x^0}$ be the strategy profile after applying the defined perturbation and $\mathbf{x^1}$ be the strategy profile that results after one round of best responses.
Consider any $i \notin S_p(\mathbf{x})$. $\bar{x}^0_{ip} = |N_i(g) \cap S_p(\mathbf{x})| x^*_p + \bar{\epsilon}_{ip}$. This in view of the fact that $|\bar{\epsilon}_{ip}| < n\delta_p < x^*_p$ implies that $b_{ip}(\mathbf{x}+\epsilon) = 0$. Further, for any $i \in S_p(\mathbf{x})$ we have $\bar{x}^0_{ip} = \bar{\epsilon}_{ip}$. Therefore $b_{ip}(\mathbf{x}+\epsilon) = x^*_p - \bar{\epsilon}_{ip}$.

Again consider any $i \notin S_p(\mathbf{x})$. $\bar{x}^1_{ip} = |N_i(g) \cap S_p(\mathbf{x})| x^*_p - \sum_{l \in N_i(g) \cap S_p(\mathbf{x})} \bar{\epsilon}_{lp}$. This in view of the fact that $|\sum_{l \in N_i(g) \cap S_p(\mathbf{x})} \bar{\epsilon}_{lp}| < n^2\delta_p < x^*_p$ implies that $b^2_{ip}(\mathbf{x}+\epsilon) = 0$. Further, for any $i \in S_p(\mathbf{x})$ we have $\bar{x}^1_{ip} = 0$. Therefore $b^2_{ip}(\mathbf{x}+\epsilon) = x^*_p$.
Thus $\mathbf{B}^2(\mathbf{x}+\epsilon) = \mathbf{x}+\epsilon$ and hence $\mathbf{B}^k(\mathbf{x}+\epsilon) = \mathbf{x}$ for any $k \geq 2$. □

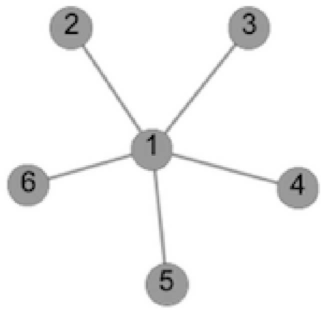
Since specialized equilibria are the only stable equilibria, we now focus our attention on specialized equilibria and investigate the existence of specialized equilibrium profiles with limited degrees of concentration of specialization. We identify necessary and sufficient conditions for the existence of two kinds of specialized Nash equilibria - (i) where concentration of specialization is limited by a maximum number of goods that any agent may specialize in, and (ii) where concentration of specialization is limited by ensuring that every agent specializes in at least one good. Propositions 5 and 6 present necessary and sufficient conditions for the first kind of equilibria, and propositions 7 and 8 do so for the second. For this section, we first present the statement of the propositions, followed by an illustrative example, before proceeding to present the formal proof. We begin by seeking, for any graph g, a necessary condition for the existence of a Q-specialized Nash equilibrium where no one specializes in more than k goods. Proposition

5 discusses this necessary condition.

Proposition 5 If $\mathbf{x}$ is a Q-specialized Nash equilibrium with $q_i \leq k$ for all $i \in N$ then there exists a collection of μ distinct maximal independent sets of g such that $\mu \geq \lceil \frac{q}{k} \rceil$. If $k = 1$ then these maximal independent sets should be mutually exclusive.

This means that if all collections of distinct maximal independent sets of g have cardinality less than $\lceil \frac{q}{k} \rceil$, then there does not exist a Q-specialized Nash equilibrium where every agent specializes in no more than k goods.

Example Consider a network g over $N = \{1, 2, 3, 4, 5, 6\}$ represented by the graph below. $I(g) = \big\{\{1\}, \{2, 3, 4, 5, 6\}\big\}$.



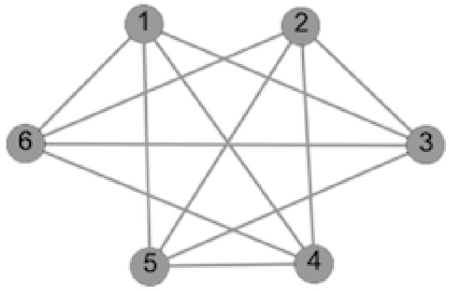
Suppose we want to identify an equilibrium which is specialized in exactly five goods, and no agent specializes in more than two goods. According to proposition 5, a necessary condition for such an equilibrium to exist is that there must be a collection of at least three $\big(\lceil \frac{5}{2} \rceil\big)$ distinct maximal independent sets of g. We observe that there are only two maximal independent sets of g, and verify that such a Nash equilibrium profile does not exist, since each of these maximal independent sets can specialize in only two goods. We now present the formal proof for proposition 5. □

Proof of Proposition 5: Let $\mathbf{x}$ be a Q-specialized Nash equilibrium with $q_i \leq k$ for all $i \in N$. Let S be the collection of sets of specialists $S = \{S_p(\mathbf{x}) | p \in Q\}$. From proposition 1, $S_p(\mathbf{x}) \in I(g)$ for every $p \in Q$. This implies that for any distinct $p, r \in Q$, $S_p(\mathbf{x}) \not\subset S_r(\mathbf{x})$. Also, $S \subseteq I(g)$.
Suppose $|S| = \mu$. The *smallest* value of μ is observed when every maximal independent set in this collection corresponds to a group of agents who specialize in the same goods. $q_i \leq k$ for every i implies that $\mu \geq \lceil \frac{q}{k} \rceil$, *i.e.* at the minimum, these distinct maximal independent sets must be equal to the smallest integer greater than or equal to q/k.
Further $k = 1$ implies that for any distinct $p, r \in Q$, $S_p(\mathbf{x}) \cap S_r(\mathbf{x}) = \phi$, so we must have at least q distinct maximal independent sets of specialists that are mutually exclusive. □

Proposition 5 shows that restricting the number of goods that any agent can specialize in places increased demands on the network structure for specialized equilibria to exist. The fewer goods that agents can specialize in, the larger is the number of distinct maximal independent sets that is needed for specialized equilibria to exist. While proposition 5 identifies a necessary condition, we now establish a sufficient condition for the existence of specialized equilibria where no one specializes in more than k goods. Proposition 6 discusses this sufficient condition.

Proposition 6 If there exists a collection of μ mutually exclusive maximal independent sets of g such that $\mu \geq \lceil \frac{m}{k} \rceil$ then there is an M-specialized Nash equilibrium with $q_i \leq k$ for all $i \in N$.

Example Suppose $M = \{1, 2, 3, 4, 5\}$. Consider a network g over $N = \{1, 2, 3, 4, 5, 6\}$ represented by the graph below. Then, $I(g) = \big\{\{1, 2\}, \{3, 4\}, \{5, 6\}\big\}$. Note that the maximal independent sets of g are all mutually exclusive.



Then, according to proposition 6, there must exist an M-specialized equilibrium in which every agent specializes in at most 2 goods. We identify one such Nash equilibrium, where, $S_1(\mathbf{x}) = S_2(\mathbf{x}) = \{1, 2\}$, $S_3(\mathbf{x}) = S_4(\mathbf{x}) = \{3, 4\}$, and $S_5(\mathbf{x}) = \{5, 6\}$. □

Proof of Proposition 6: Consider $C = \{I_1, I_2, \ldots, I_\mu\} \subseteq I(g)$ where $I_s \cap I_t = \phi$ for any distinct $I_s, I_t \in C$ and $\mu \geq \lceil \frac{m}{k} \rceil$. Consider a strategy profile $\mathbf{x}$ such that $(\forall r \in \{1, \ldots, \mu\})(\forall i \in I_r)[(\forall p \in \{(r-1)k+1, \ldots, \min\{m, rk\}\})(x_{ip} = x_p^*) \wedge (\forall p \notin \{(r-1)k+1, \ldots, \min\{m, rk\}\})(x_{ip} = 0)]$ and $(\forall i \notin \cup I_r)(\forall p \in M)(x_{ip} = 0)$.

That is, agents in I_1 specialize in goods $1, \ldots k$ (and make zero investment in all other goods), agents in I_2 specialize in goods $k+1, \ldots 2k$ (and make zero investment in all other goods), and so on, till every good in M has a set of specialists. Further, any agent not in $\cup I_r$ makes zero investment in every good. From proposition 1, it is immediate that $\mathbf{x}$ is a M-specialized Nash equilibrium. Further, mutual exclusivity of sets of specialists ensures that every agent specializes in no more than k goods. Thus $\mathbf{x}$ is a M-specialized Nash equilibrium with $q_i \leq k$ for all $i \in N$. If $\mu = m$ and I is a partition of N then $\mathbf{x}$ is a M-specialized Nash equilibrium with $q_i = 1$ for all $i \in N$. □

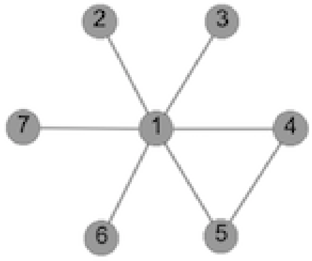
Thus the existence of a certain number of mutually exclusive maximal independent sets can allow for existence of M-specialized Nash equilibria with related concentration of specialization. For instance, if the number of mutually exclusive maximal independent sets in a graph is low, then only highly concentrated M-specialized Nash equilibria exist, wherein specialists specialize in a large number of goods. Both propositions 5 and 6 establish this relation between the network structure and concentration of specialized equilibria.

We now turn our attention to limiting concentration of specialization by focusing on equilibrium profiles in which no one free-rides completely, and every agent specializes in at least one good. We seek, for any graph g, a necessary condition for the existence of a Q-specialized Nash equilibrium where everyone specializes in at least one good. Proposition 7 discusses this necessary condition.

Proposition 7 If $\mathbf{x}$ is a Q-specialized Nash equilibrium with $q_i \geq 1$ for all $i \in N$ then there exists $C \subseteq I(g)$ such that $\cup C = N$ and $|C| \leq q$. If $\mathbf{x}$ is a Q-specialized Nash equilibrium with $q_i = 1$ for all $i \in N$, then there exists a collection of q maximal independent set of g which partitions N.

This means that if there exists a Q-specialized Nash equilibrium in which everyone specializes in atleast one good, then there must exist a collection of maximal independent sets of g which has at most q elements/sets, and the union of this collection of sets equals N.

Example Consider a network g over $N = \{1, 2, 3, 4, 5, 6, 7\}$ represented by the graph below. $I(g) = \big\{\{1\}, \{2, 3, 4, 6, 7\}, \{2, 3, 5, 6, 7\big\}$.



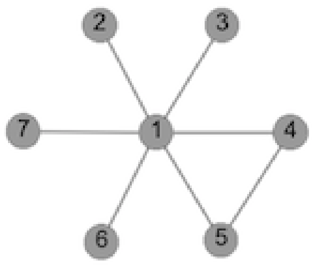
Suppose we want to identify a Nash equilibrium that is specialized in exactly two goods, with every agent specializing in atleast one good. According to proposition 6, for such an equilibrium to exist, there must exist a collection of at most two maximal independent sets such that their union equals N. In this case, $I(g)$ is also the only collection of maximal independent sets of g for which the union equals N, and has three elements/sets. Since every agent must specialize in at least one good, and every set of specialists corresponds to a maximal independent set of g, we can verify that a Nash equilibrium specialized in exactly two goods not exist.

Proof of Proposition 7: Let $\mathbf{x}$ be a Q-specialized Nash equilibrium with $q_i \geq 1$ for all $i \in N$. This implies that
(i) For any $p \in Q$, $S_p(\mathbf{x}) \neq \phi$ (since $\mathbf{x}$ is Q-specialized)
(ii) $S_p(\mathbf{x}) \in I(g)$ for all $p \in Q$ (from proposition 1).
(iii) $(\forall i \in N)(\exists p \in Q)(i \in S_p(\mathbf{x}))$ (since $q_i \geq 1$ for all i) and therefore $\underset{p\in Q}{\cup} S_p(\mathbf{x}) = N$.
Thus we have a collection of non empty maximal independent sets the union of which equals N. The number of these sets can be at most equal to q because every agent must specialize in atleast one good ($q_i \geq 1$). Further, if $q_i = 1$ for all $i \in N$, then $(\forall p, r \in Q)(S_p(\mathbf{x}) \cap S_r(\mathbf{x}) = \phi)$ for $p \neq r$, and these sets are mutually exclusive, hence partitioning N. □

We now identify a condition sufficient for the existence of specialized Nash equilibria where every agent specializes in at least one good exist.

Proposition 8 If there exists a collection of maximal independent sets $I \subseteq I(g)$ such that $|I| \leq m$ and $\cup I = N$ then there is an M-specialized Nash equilibrium with $q_i \geq 1$ for all $i \in N$. If there exists a collection of m maximal independent set of g which partitions N then there is an M-specialized Nash equilibrium with $q_i = 1$ for all $i \in N$.

Example Suppose $M = \{1, 2, 3, 4, 5\}$. Consider a network g over $N = \{1, 2, 3, 4, 5, 6, 7\}$ represented by the graph below. $I(g) = \big\{\{1\}, \{2, 3, 4, 6, 7\}, \{2, 3, 5, 6, 7\}\big\}$.



Note that $I(g)$ is a collection of maximal independent sets of g for which the union equals N. This collection has less than 5 sets, and from proposition 8, there must exist an M-specialized equilibrium in which every agent specializes in at least one good. We identify one such specialized strategy profile with $S_1(\mathbf{x}) = S_4(\mathbf{x}) = S_5(\mathbf{x}) = \{1\}$, $S_2(\mathbf{x}) = \{2, 3, 4, 6, 7\}$ and $S_3(\mathbf{x}) = \{2, 3, 5, 6, 7\}$. This profile is an M-specialized Nash equilibrium. □

Proof of Proposition 8: Suppose $I = \{I_1, I_2, \ldots, I_\mu\} \subseteq I(g)$, $\cup I = N$ and $\mu \leq m$. Consider a specialized strategy profile $\mathbf{x}$ such that $(\forall i \in I_1)(\forall p \in \{1, \mu + 1, \ldots, m\})(x_{ip} = x_p^*)$ and

$(\forall p \in \{2, \dots, \mu\})(\forall i \in I_p)(x_{ip} = x_p^*)$.
It is immediate that $\mathbf{x}$ is an M-specialized Nash equilibrium with $q_i \geq 1$ for all $i \in N$.
If $\mu = m$ and I is a partition of N then $\mathbf{x}$ is a M-specialized Nash equilibrium with $q_i = 1$ for all $i \in N$.

4. Welfare analysis

The total welfare generated from strategy profile $\mathbf{x}$ on graph g is given by

$$W(\mathbf{x}, g) = \sum_{i \in N} U_i(\mathbf{x}, g)$$
$$= \sum_{i \in N} \left[F(x_{i1} + \overline{x}_{i1}, x_{i2} + \overline{x}_{i2}, \dots x_{im} + \overline{x}_{im}) - \sum_{p \in M} c_p x_{ip}\right]$$

Following a utilitarian approach, strategy profile $\mathbf{x}$ is *efficient* for a given network g if and only if there does not exist any other profile $\mathbf{x'}$ such that $W(\mathbf{x'}, g) > W(\mathbf{x}, g)$. Because the utility function is additively separable in all goods (second order cross partial derivatives are zero), the welfare generated by any strategy profile over a given graph is a simple aggregate of the welfare generated from investment on each good. Since the welfare function is concave, in an efficient profile the following must hold *for every good p in M*:

$$\big(\forall i \in N\big)\big(x_{ip} > 0 \longrightarrow \frac{\delta W(\mathbf{x}, g)}{\delta x_{ip}} = 0\big) \text{ and } \big(\forall i \in N\big)\big(x_{ip} = 0 \longrightarrow \frac{\delta W(\mathbf{x}, g)}{\delta x_{ip}} \leq 0\big)$$

That is, for any agent i who invests a positive amount in good p, it must be that

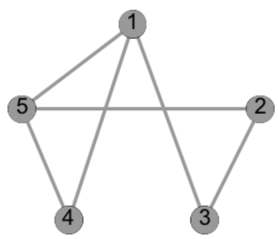
$$F_p(x_{ip} + \bar{x}_{ip}) + \sum_{j \in N_i(g)} F_p(x_{jp} + \bar{x}_{jp}) = c_p \qquad (1)$$

where the left hand side is the marginal social benefit from i's investment in good p, and this must equal marginal cost of good p. If i's investment is zero in the efficient profile, this marginal social benefit must be at most equal to the marginal cost. Thus the efficient or welfare maximising strategy profile is one where agents invest in every single good in a way that maximises welfare from that good.

From lemma 1, we have deduced that in any Nash equilibrium profile, if any agent makes positive investment in any good, then her marginal benefit from the good must equal marginal cost. That is, for any agent i who invests a positive amount in good p, it must be that $F_p(x_{ip} + \bar{x}_{ip}) = c_p$. Comparing this with (1), we immediately see that Nash equilibria are not efficient because efficiency requires agents to consider not only individual marginal benefit, but also the positive externalities of their investment when making investment decisions on various goods.

When we consider the welfare consequences of limiting concentration of specialization in specialized equilibria, we find that this may be detrimental to welfare. Consider the following example.

Example $M = \{1, 2\}$, $F(x_1, x_2) = x_1^{0.5} + x_2^{0.5}$ and $c_1 = c_2 = 0.5$, such that $x_1^* = x_2^* = 1$. Consider a network g over $N = \{1, 2, 3, 4, 5\}$ represented by the graph below. $I(g) = \{\{1, 2\}, \{3, 4\}, \{2, 4\}, \{3, 5\}\}$.



In this network, all maximal independent sets g the same cardinality. We consider two M-specialized equilibrium profiles, $\mathbf{e_1}$ where $S_1(\mathbf{e_1}) = \{1, 2\}$ and $S_2(\mathbf{e_1}) = \{3, 4\}$, and $\mathbf{e_2}$ where $S_1(\mathbf{e_2}) = S_2(\mathbf{e_2}) = \{1, 2\}$. Note that in both these equilibria, since the number of specialists for each good is the same, the aggregate cost of investment is equal. However, the distribution of access to these investments, and hence utilities, differs across the two equilibria. The total welfare associated with $\mathbf{e_1}$ and $\mathbf{e_2}$ is calculated using the tables 1 and 2 respectively.

Table 1. Investment levels, access and costs in e_1

i	$x_{i1} + \overline{x}_{i1}$	$x_{i2} + \overline{x}_{i2}$	$c_1 x_{i1} + c_2 x_{i2}$
1	1	2	0.5
2	1	1	0.5
3	2	1	0.5
4	1	1	0.5
5	2	1	0

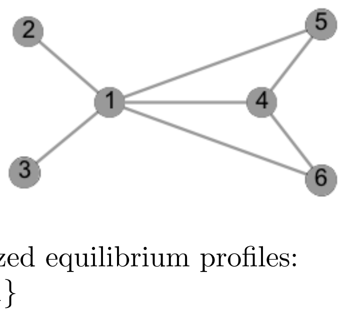
Table 2. Investment levels, access and costs in e_2

i	$x_{i1} + \overline{x}_{i1}$	$x_{i2} + \overline{x}_{i2}$	$c_1 x_{i1} + c_2 x_{i2}$
1	1	1	1
2	1	1	1
3	2	2	0
4	1	1	0
5	2	2	0

From these tables, we calculate that $W(\mathbf{e_1}, g) = 3\sqrt{2} + 5 \simeq 9.24$ and $W(\mathbf{e_2}, g) = 4\sqrt{2} + 4 \simeq 9.66$. Thus in this example we find that of the two specialized equilibria, the profile with higher degree of concentration of specialization also has higher welfare. This is driven by the fact that investments made by agents in the maximal independent set $\{1, 2\}$ are more accessed by other players than investments made by agents in the maximal independent set $\{3, 4\}$. This shows that in absence of limits on degree of concentration of specialization, in some cases there may be welfare advantages driven by the fact that some maximal independent sets, in the role of specialists, yield more positive externalities than others. Consequently, limiting the degree of concentration of specialization *may* have negative welfare consequences.

On the other hand, we also have scenarios where high concentration of specialization has negative welfare consequences associated. The next example illustrates one such scenarios.

Example $M = \{1, 2\}$, $F(x_1, x_2) = x_1^{0.2} + x_2^{0.8}$ and $c_1 = 0.2$ and $c_2 = 0.8$, such that $x_1^* = x_2^* = 1$. Consider a network g over $N = \{1, 2, 3, 4, 5, 6\}$ represented by the graph below. Here, $I(g) = \{\{1\}, \{2, 3, 4\}, \{2, 3, 5, 6\}\}$ such that all maximal independent sets have different cardinalities.



We first consider four M-specialized equilibrium profiles:

$\mathbf{e_1}$ where $S_1(\mathbf{e_1}) = S_2(\mathbf{e_1}) = \{1\}$
$\mathbf{e_2}$ where $S_1(\mathbf{e_2}) = S_2(\mathbf{e_2}) = \{2, 3, 4\}$
$\mathbf{e_3}$ where $S_1(\mathbf{e_3}) = S_2(\mathbf{e_3}) = \{2, 3, 5, 6\}$
$\mathbf{e_4}$ where $S_1(\mathbf{e_4}) = \{1\}$ and $S_2(\mathbf{e_4}) = \{2, 3, 5, 6\}$

The total welfare associated with $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, and $\mathbf{e_4}$, is calculated using the tables 3, 4, 5, and 6 respectively.

$W(\mathbf{e_1}, g) = 12 - 1 = 11$
$W(\mathbf{e_2}, g) = 10 + 3^{0.2} + 3^{0.8} - 3 = 10.65$
$W(\mathbf{e_3}, g) = 8 + 2^{0.2} + 4^{0.2} + 2^{0.8} + 4^{0.8} - 4 = 11.24$
$W(\mathbf{e_4}, g) = 10 + 4^{0.8} + 2^{0.8} - 3.4 = 11.371$

Table 3. Investment levels, access and costs in e_1

i	$x_{i1} + \overline{x}_{i1}$	$x_{i2} + \overline{x}_{i2}$	$c_1 x_{i1} + c_2 x_{i2}$
1	1	1	1
2	1	1	0
3	1	1	0
4	1	1	0
5	1	1	0
6	1	1	0

Table 4. Investment levels, access and costs in e_2

i	$x_{i1} + \overline{x}_{i1}$	$x_{i2} + \overline{x}_{i2}$	$c_1 x_{i1} + c_2 x_{i2}$
1	3	3	0
2	1	1	1
3	1	1	1
4	1	1	1
5	1	1	0
6	1	1	0

Table 5. Investment levels, access and costs in e_3

i	$x_{i1} + \overline{x}_{i1}$	$x_{i2} + \overline{x}_{i2}$	$c_1 x_{i1} + c_2 x_{i2}$
1	4	4	0
2	1	1	1
3	1	1	1
4	2	2	0
5	1	1	1
6	1	1	1

Table 6. Investment levels, access and costs in e_4

i	$x_{i1} + \overline{x}_{i1}$	$x_{i2} + \overline{x}_{i2}$	$c_1 x_{i1} + c_2 x_{i2}$
1	1	4	0.2
2	1	1	0.8
3	1	1	0.8
4	1	2	0
5	1	1	0.8
6	1	1	0.8

First we compare specialized equilibria in which a single set of specialists specializes in both goods, i.e., equilibria $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$. Among these, $\mathbf{e_1}$ has the highest concentration of specialization by all measures, with a single agent specializing and all other agents free-riding in both goods. In comparison, $\mathbf{e_2}$ and $\mathbf{e_3}$ have lower concentration of specialization, since fewer people free-ride in these equilibria. On comparing welfare levels across these three profiles, we find that $W(\mathbf{e_3}, g) > W(\mathbf{e_1}, g) > W(\mathbf{e_2}, g)$. That is, while one of the less concentrated equilibria has greater welfare than $\mathbf{e_1}$, the other has lower welfare. The differences in welfare levels across the three equilibria are driven not just by the number of specialists in each case, but also by the fact that some specialists are better placed than others to yield positive externalities from investment that more than justify the costs. This is seen more clearly by comparing equilibrium $\mathbf{e_3}$ with the lesser concentrated equilibrium $\mathbf{e_4}$ to observe that $W(\mathbf{e_4}, g) > W(\mathbf{e_3}, g)$. This higher welfare in $\mathbf{e_4}$ is driven by the fact that *given the shape of the utility function and the marginal costs of investment for the two goods*, the maximal independent set $\{1\}$ is best placed to specialize in good 1, and the maximal independent set $\{2, 3, 4, 5\}$ is best placed to specialize in good 2. This is verified by the fact that specialized equilibrium $\mathbf{e_4}$ has the highest welfare level among all specialized equilibrium profiles that are possible[2] over the given network.

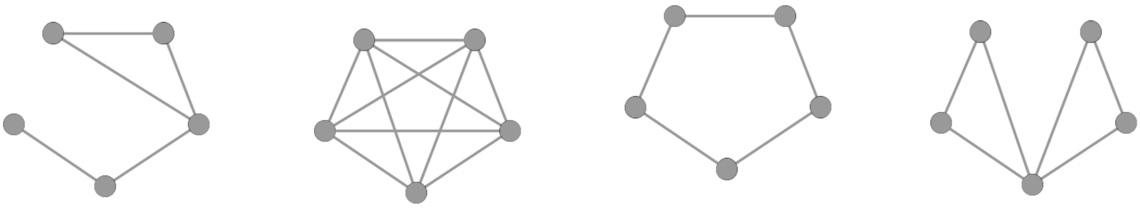
This example illustrates the possibilities of welfare advantages associated with specialized equilibria with limited concentration of specialization, driven by the differences in marginal utilities and costs of the various goods and the fact that different maximal independent sets are best suited to specialize in different goods.

5. Conclusion

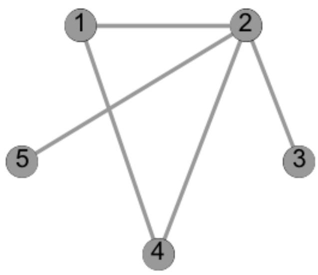
In the context of multiple goods that are non-excludable and non-rivalrous across links in a fixed network, we find that specialized Nash equilibria are the only stable equilibria. Our search for specialized Nash equilibria with limited degrees of concentration is motivated by the unsustainability of highly concentrated equilibria as suggested by the robust body of literature on public goods. Particularly, in contexts where links in a network may be broken, concerns of fairness and reciprocity become important in influencing people's decisions about investments in various goods.

While specialized equilibria with specialization concentrated over a small set of agents may be undesirable from an equity point of view, the existence of such equilibria, within our framework, is certain. This is because specialists for any good must necessarily comprise a maximal independent set of the network, and for any network structure, there always exists at least one maximal independent set. However, the existence of Nash equilibria with limited/lower degrees of concentration of specialization places additional demands on the structure of the network. For instance, when we limit concentration of specialization in M-specialized equilibria by ensuring that every agent specializes in at least one good, we find that the condition necessary and sufficient for the existence of such equilibria is the existence of a collection of *at most* m maximal independent sets that together exhaust N. The set of *all* maximal independent sets of a graph, $I(g)$, is the largest such collection that exhausts N. Since all other collections that exhaust N are subsets of $I(g)$, the cardinality of these collections is capped by the cardinality of $I(g)$. Let $h(n)$ be the maximum possible number of maximal independent sets that a graph over n nodes/agents may have. If $m > h(n)$ (the number of goods is larger than the maximum possible number of maximal independent sets), we know that $m > |I(g)|$ for any network g with n agents, and the existence of an M-specialized equilibrium where every agent specializes in at least one good is guaranteed. Literature on graph theory, beginning with Moon and Moser (1965), has explored the question of the maximum possible maximal independent sets. Füredi (1987) generalized a theorem by Moon and Moser to determine the maximum number of maximal independent sets in a connected graph on $n > 50$ nodes. Griggs et al (1988) have determined this number for connected graphs over any n. However, for certain n, if $h(n) > m$, $h(n)$ may be too large to be relevant/useful for our purpose. For example, for $n = 5$, $h(n) = 5$. This highest number of maximal independent sets materialises in the four specific network architectures, all of which are shown in figure below.

[2] Given two goods and three maximal independent sets of the graph, there are a total of nine possible specialized equilibria. We define $\mathbf{e_5}$ where $S_1(\mathbf{e_5}) = \{2, 3, 5, 6\}$ and $S_2(\mathbf{e_5}) = \{1\}$, $\mathbf{e_6}$ where $S_1(\mathbf{e_6}) = \{1\}$ and $S_2(\mathbf{e_6}) = \{2, 3, 4\}$, $\mathbf{e_7}$ where $S_1(\mathbf{e_7}) = \{2, 3, 4\}$ and $S_2(\mathbf{e_7}) = \{1\}$, $\mathbf{e_8}$ where $S_1(\mathbf{e_8}) = \{2, 3, 4\}$ and $S_2(\mathbf{e_8}) = \{2, 3, 5, 6\}$, and $\mathbf{e_9}$ where $S_1(\mathbf{e_9}) = \{2, 3, 5, 6\}$ and $S_2(\mathbf{e_9}) = \{2, 3, 4\}$. The associated welfare levels are $W(\mathbf{e_5}, g) = 10.87$, $W(\mathbf{e_6}, g) = 10.80$, $W(\mathbf{e_7}, g) = 10.84$, $W(\mathbf{e_8}, g) = 11.21$, and $W(\mathbf{e_9}, g) = 10.66$.



Over these network structures, for $m = 3$, an M-specialized equilibrium with every agent specializing in at least one good will not sustain. However, for many other network structures, this is not the case. Consider the network shown below and the sets of specialists $S_1(x) = \{1, 3, 5\}, S_2(x) = \{3, 4, 5\}, S_3(x) = \{2\}$.



For $n = 5$ and $m = 3$, the strategy profile described above is an M-specialized Nash equilibrium where all agents specialize in at least one good.

On the other hand, the condition sufficient to ensure the existence of an M-specialized Nash equilibrium where no one specializes in more than k goods is the existence of a collection of atleast $\lceil \frac{m}{k} \rceil$ *mutually exclusive* maximal sets of the graph in consideration. The existence of a collection of atleast $\lceil \frac{m}{k} \rceil$ distinct maximal sets is necessary for the existence of an M-specialized Nash equilibrium where no one specializes in more than k goods. The number of distinct maximal independent sets of a graph representing a network may be fairly low, and depending on the values of m and k, many networks may not sustain specialized equilibria with concentration of specialization limited in this manner. This makes it important to ask: how is the architecture/structure of graphs linked with the number of maximal independent sets? For any given n, which network structures accommodate the highest number of maximal independent sets? Which network structure accommodate the highest number of *mutually exclusive* maximal independent sets? These questions remain open in graph theory.

In our multiple public goods analysis, we have assumed that there is no interaction between various goods in the utility accruing to agents. While this assumption is essential to the analysis of stability of Nash equilibria as well as welfare analysis, it is not essential to our analysis of specialized Nash equilibria. As long as the utility function is concave and takes a form that allows for a identification of a *unique* vector $x^* = (x_1^*, x_2^*, \ldots, x_m^*) \in \Re_+^m$ where $x_p^* = arg[F_p(\cdot) = c_p]$ for any $p \in M$, our equilibrium analysis holds. This extension, along with a consideration of budget constraints, requires further investigation.

References

[1] Bramoullé, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic theory*, 135(1):478–494.

[2] Cherry, T. L. and Dickinson, D. L. (2007). Voluntary contributions with multiple public goods. In *Environmental economics, experimental methods*, pages 206–215. Routledge.

[3] Dawes, R. M. (1980). Social dilemmas. *Annual review of psychology*, 31(1):169–193.

[Eek et al.] Eek, D., Biel, A., and Gärling, T. *The fairness effect on cooperation in asymmetric social dilemmas when equality is perceived as unfair.*

[5] Fehr, E. and Gächter, S. (2000). Cooperation and punishment in public goods experiments. *American Economic Review*, 90(4):980–994.

[6] Füredi, Z. (1987). The number of maximal independent sets in connected graphs. *Journal of Graph Theory*, 11(4):463–470.

[7] Ghosh, S., Karaivanov, A., and Oak, M. (2007). A case for bundling public goods contributions. *Journal of Public Economic Theory*, 9(3):425–449.

[8] Griggs, J. R., Grinstead, C. M., and Guichard, D. R. (1988). The number of maximal independent sets in a connected graph. *Discrete Mathematics*, 68(2-3):211–220.

[9] Kurzban, R. and DeScioli, P. (2008). Reciprocity in groups: information-seeking in a public goods game. *European Journal of Social Psychology*, 38(1):139–158.

[10] Maier-Rigaud, F. P., Martinsson, P., and Staffiero, G. (2010). Ostracism and the provision of a public good: experimental evidence. *Journal of Economic Behavior & Organization*, 73(3):387–395.

[11] Marwell, G. and Ames, R. E. (1979). Experiments on the provision of public goods. i. resources, interest, group size, and the free-rider problem. *American Journal of sociology*, 84(6):1335–1360.

[12] Moon, J. W. and Moser, L. (1965). On cliques in graphs. *Israel journal of Mathematics*, 3(1):23–28.

[13] Mutuswami, S. and Winter, E. (2004). Efficient mechanisms for multiple public goods. *Journal of Public Economics*, 88(3-4):629–644.

[14] Ostrom, E., Walker, J., and Gardner, R. (1992). Covenants with and without a sword: Self-governance is possible. *American political science Review*, 86(2):404–417.

[15] Richefort, L. (2018). Warm-glow giving in networks with multiple public goods. *International Journal of Game Theory*, 47(4):1211–1238.

[16] van Dijk, E. and Wilke, H. (1984). Conditional contributions and public good provision. In *Social dilemmas and cooperation*, pages 173–185. Springer.

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