The Second Fundamental Theorem of Positive Economics

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Abstract

Welfare Economics is fortunate that there are two Fundamental Theorems of Welfare Economics. Positive Economics on the other hand is seemingly endowed with none. One of the fundamental results of Positive Economics is that a competitive equilibrium exists under fairly general conditions; this then may be called the First Fundamental Theorem of Positive Economics (FFTPE). The existing results on uniqueness and stability of competitive equilibrium are far too restrictive to be up for consideration as a Fundamental Theorem. It is to re-examine this question that we revisit the question of stability of competitive equilibrium. It is shown that if, for all distributions of the aggregate endowment, the matrix sum of the Jacobian of the excess demand function plus its transpose, evaluated at the equilibrium, have maximal rank then equilibria will be locally asymptotically stable. When this condition is not met, it is shown how redistributing resources will always make a competitive equilibrium price configuration stable and this need not involve redistributing endowments so that trades do not exist at equilibrium. This last result is quite general and the only requirement is that the rank condition referred to earlier hold at zero trade competitive equilibria and consequently may qualify to be called the Second Fundamental Theorem of Positive Economics (SFTPE).

Keywords

Stability of equilibrium, redistribution of resources, rank condition, Fundamental Theorems.

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1 Introduction

In our search for what may be called a Fundamental Theorem of Positive Economics, the first result which should be considered is the existence of a Competitive Equilibrium under fairly general conditions. The competitive market was supposed to solve for the equilibrium prices by itself. In fact the famous `Invisible Hand' was supposed to be able to achieve this and this conclusion is mistakenly attributed to Adam Smith. However, it is only later writers who have attributed this power to the Invisible Hand[1]. Not only were writers wrong about the source of this belief, they appear to have been mistaken in their belief that the Invisible Hand was successful in attaining an equilibrium. We shall be concerned with the latter aspect in this paper. As we have remarked, when subjected to scrutiny, this belief in the Invisible Hands power, did not hold up and conditions under which this was possible, the so-called `stability conditions' needed to be invoked. The working of the Invisible Hand was through the forces of demand and supply, it may be recalled; consequently the need for stability conditions implied that demand and supply did not possess the power to achieve this target. That stability of market economy (or the ability of the market to solve for the equilibrium prices through the forces of demand and supply) could not be taken for granted was first noted by Scarf (1960) and Gale (1963). Their exercise consisted of setting up a class-room type example of a market economy: a one market economy involving two persons and a two market economy with three persons in the case of latter; specification of tastes and resources available led to the construction of demand functions; supplies were assumed fixed since what was being studied was just the exchange process. And it was found that the price adjustment in the direction of excess demand (i.e., demand minus supply)[2] need not necessarily lead to the equilibrium. Thus stability, it was implied, was a special

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[1] Since the term Invisible Hand was thought to be coined by Adam Smith in the celebrated book Wealth of Nations, the role of the Invisible Hand in equilibrating markets is some times attributed to Adam Smith; but Smith mentions Invisible Hand once in History of Astronomy, for the first time, completed around 1758 and then in his book The Theory of Moral Sentiments (1759) and then in the Wealth of Nations (1776). In addition, the reference in the last was made not while discussing markets in Books I and II but only in Book IV where Smith was advocating support of domestic industry over foreign! So while we use the term Invisible Hand, it should be noted that the failures or successes of this instrument should not be attributed to Adam Smith but rather to those who thought that Smith said so and followed this bit of fiction blindly. For a more modern look at the Invisible Hand, see Billot (2009).

[2] We are using a simple form of these equations where the price change is proportional to the level of excess demand and the constant of proportionality is unity. This simplifies exposition considerably and choosing the factor to be unity is not of significance. What is a significant restriction is to choose the price adjustment to be proportional to excess demand; the intuition is basically that the rate of price change in any market should have merely the same

property. Around the same time as these examples were being investigated, work was also progressing on another front: namely finding out conditions on excess demand functions which led to stability that is identifying stability conditions. Basically these stability conditions were in the nature of restriction on preferences or tastes of decision makers or agents: See for example, Negishi (1960) and Hahn (1982) for surveys of this area. It was noted too that had the market demand originated from the maximization of a single welfare function or if tastes were similar to the extent that net buyers and net sellers behaved similarly, stability of equilibrium could be ensured. It would therefore appear that if preferences were diverse, which is the setting for this exercise, these conditions may be difficult to ensure. And consequently, to ensure stability, we could no longer rely on preferences being restricted in some manner. Thus alternative avenues needed to be explored. Indeed as we shall see, without restricting preferences in any manner, one may still obtain stability of equilibrium by redistributing resources. The implications of the necessity for such a course of action may not be evident immediately and we shall return to this later. For the moment, we investigate this phenomenon in some detail. And it is this investigation which shall lead to what we suggest may be called the Second Fundamental Theorem of Positive Economics.

2 Example of Instability

2.1 The Gale Example

Consider the following example due to Gale (1963). There are two persons **A,B** with utility functions defined over commodities (x,y) as follows: $U_A(x,y) = \min(x,2y)$ and $U_B(x,y) = \min(2x,y)$; their endowments are specified by $w_A = (1,0)$, $w_B = (0,1)$; routine computations lead to the excess demand function of the first good (x), $Z(p)$, for $p > 0$, where p is the relative price of good x:

$$Z(p) = \frac{p-1}{p+2 \ (2p+1)}$$

Thus the unique **interior** equilibrium is given by $p = 1$[3]; now notice that if the adjustment on prices is given by

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sign as excess demand. See however Mukherji (2008) on the justification for choosing the rate of price adjustment to be a constant proportion of the excess demand.

[3] There are two other equilibria: equilibrium at infinity and an equilibrium at 0. The equilibrium at infinity follows since $lim_{p\to\infty} Z(p) = 0$. The equilibrium at p = 0 has A consuming the bundle $(x_A, 0)$ and B consuming $(x_B, 1)$ where $x_A + x_B = 1$, $x_A, x_B \geq 0$, $x_B \geq 1/2$; further this equilibrium is locally asymptotically stable. To see how there is an

$$p \;= \mathrm{h(p)} \tag{1}$$

where $\mathrm{h(p)}$ has the same sign as $\mathrm{Z\,(p)}$ and is continuously differentiable so that the solution to (1) say $p_t\ (p^o)$ is well defined for any initial point p^o > 0.

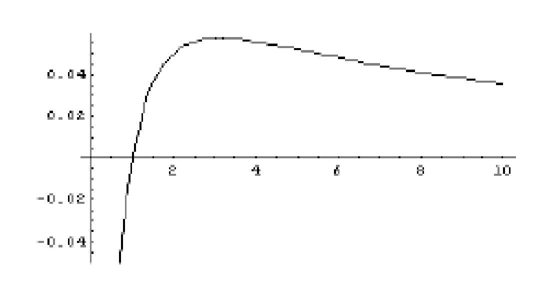

Figure 1: Excess Demand - The Gale Example

As Gale (1963)[4] says, "Arrow and Hurwicz have shown that for the case of two goods, one always has global stability...... Nevertheless, some queer things can happen even in this case." To see the queer things referred to, consider the function $\mathrm{V}\,(p_t)$ = $(p_t - 1)^2$ and notice that along the solution to the equation (1), we have $\mathrm{V\,(t)}$ > 0 for all t, if $p^o \neq 1$: so that the price moves further away from equilibrium and there is no tendency to approach the unique interior equilibrium.

Notice that the excess demand curve is upward rising at the interior equilibrium and hence we have the above conclusion. However, in this set up, let us tinker with the distribution of resources. Suppose for example, we interchange the endowments i.e., **A** has (0,1) while **B** has (1,0). One may note that at equilibrium $p^* = 1$, the purchasing power has remained the same and hence so do the demands but because endowments have changed the trades at equilibrium are different. Recomputing excess demand functions, we note that the unique interior equilibrium is now globally stable. This follows since the excess demand function, for $\mathrm{p} > 0$, is now given by:

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equilibrium at p = 0, notice that at p = 0, the demand by A is any member of the set $x, 0$ such that $0 \le x \le 1$; while B's demand is any member of the set $(x, 1)\ :\ x \ge 1/2$; hence the claim follows. That p = 0 is locally asymptotically stable follows from the Figure 1.

[4] There are two sets of examples in this contribution; we consider here the two-good example. A treatment of the three good example is contained in Mukherji (1973); see also Bala (1997), in this connection.

$$Z(p) = \frac{2(1-p)}{(2p+1)\ p+2}$$

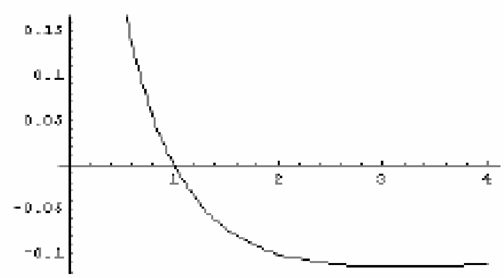

Figure 2: Gale Example with a switch in endowments

Notice now that the instability of the interior equilibrium noted earlier disappears. One may therefore say that we had instability of the interior equilibrium because the pattern of purchasing power, in relation to endowments had not been *right*. With the new pattern of endowments, excess demand curve becomes downward sloping. This should be the first indicator that for stability, an appropriate distribution of endowments may be essential. Notice too that this is necessary because individuals are not identical in either tastes or endowments and this is why such investigations assume importance.

It may be instructive to consider the Gale example in some further detail. We _first considered the endowment distribution in Gale: (1,0), (0,1) for *A, B* respectively; we then switched it to (0, 1) (1,0) for *A, B* respectively. Consider a weighted average of these two distributions $(\lambda, 1-\lambda)$, $(1-\lambda, \lambda)$ to *A, B* respectively, where $(0 \leq \lambda \leq 1)$; thus for $\lambda = 1$, we have the Gale endowment pattern and for $\lambda = 0$ we have the switched pattern that we used to deduce Figure 2; notice that at $p = 1$ the purchasing power of the individuals remains the same at these distributions; consequently the demand does not change and hence $p = 1$ is an equilibrium for each such distribution; however the excess demand function changes. Routine calculations yield:

$$f(p, \lambda) \equiv Z_x = \frac{2(\lambda\ p-1\ +1)}{(2p+1)} + \frac{p+\lambda\ 1-p\ +1)}{(p+2)} - 1$$

Consequently

$$Z_x = \frac{p-1\ \ (3\lambda-2)}{2p+1\ \ (p+2)}$$

and hence

$$\text{Sign of } Z_{xp}|_{p=1} = \text{Sign of } (3\lambda - 2);$$

hence $p = 1$ for all values of $\lambda < 2/3$ is **stable**; when $\lambda = 2/3$, the derivative vanishes (in fact, $Z_x\ \ p\ \ = 0 \forall p$ if $\lambda = 2/3$).. Our choice of $\lambda = 0$ worked to stabilize the equilibrium but clearly as is evident, there are many other possible redistributions which will achieve the same end. The following diagram may clarify how changes in the values of λ alters the excess demand function.

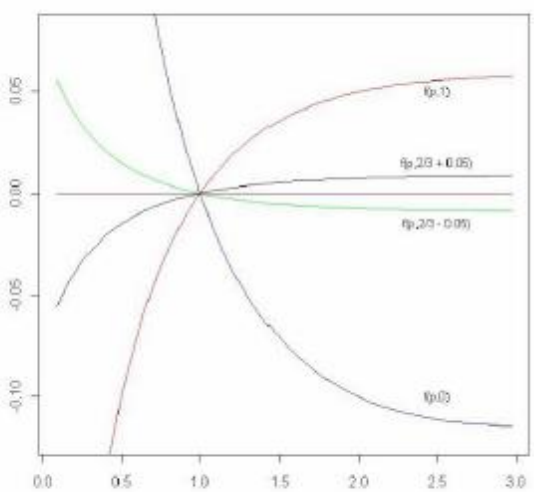

Figure 3: Excess Demands for alternative values of λ

Notice that the excess demands $f(p, 0)$ and $f\ \ p, 1$ were drawn earlier; $f(p, 2/3)$ is a horizontal through the point (0, 0); if $\lambda > 2/3$ the excess demand is downward sloping at $p = 1$ while for $\lambda < 2/3$ the excess demand is upward sloping at $p = 1$. Thus there are many endowment distributions which would render the interior equilibrium stable. We examine below how general this inference is.

3 The Model

We shall assume as in Negishi (1962) that we are analyzing the standard exchange model involving m individuals and n goods and that the total amounts of these goods is given by the components of $W \in \Re^n_{++}$; each individual has a real-valued utility function $U^i: \Re^n_+ \rightarrow \Re$; further each U^i is assumed to be *strictly increasing, strictly quasi-concave and continuously differentiable*.

Sometimes, we shall specify a distribution of W among the individuals usually denoted by $w^i \in \Re^n_{++}$ such that $\sum_i w^i \leq W$; let us denote the set of all such feasible allocations by the set $\mathcal{W}$; if an allocation $\{w^i\}$ has been chosen from $\mathcal{W}$, we can then proceed with defining demands $x^i(P)$ as the unique maximizer of U^i in the budget set provided by[5] $\{x: P^T.x \leq P^T.w^i, x \geq 0\}$ where $P \in \Re^n_{++}, P = (p_1, \ldots, p_n)$ is the price vector; **in case** we have a numeraire, we shall consider good n to be the numeraire and write the price vector as $P = (p, 1)$; the vector of **relative prices** will then be written as $p \in \Re^{n-1}_{++}$.

Market demands are defined by $X(P) = \sum_i x^i(P)$; excess demand is then defined by $Z(P) = X(P) - W$. Strictly speaking we should write $Z(P, \{w^i\})$ however, we usually omit the distribution of the resources and write $Z(P)$.

Excess demand functions are expected to satisfy:

1. $Z(P)$ is a continuous function and bounded below for all P > 0;
2. Homogeneity of degree zero in the prices i.e., $Z(\theta P) = Z(P)\ \forall \theta > 0, P > 0$.
3. Walras Law i.e., $P^T.Z(P) = 0\ \forall\ P > 0$;

to these we add the following assumptions:

4. $Z(P)$ is twice continuously differentiable function of prices for all $P > 0$.
5. For any sequence, $P^s = (P_1^s, P_2^s, \ldots\ldots, P_n^s) \in \Re^n_{++}, P_i^s = 1, \forall s$ for some index i, say $i = i_0$ and $\|P^s\| \to +\infty$ as $s \to +\infty \Rightarrow \|Z_{i_0}(P^s)\| \to +\infty$[6] (Boundary condition).

The above conditions are standard and all of them excluding the last, in fact appeared in Negishi (1962); the importance of the role of assumptions such as the last, (the Boundary condition), was realized somewhat later[7]. Finally, the equilibrium for the economy, with individual resources $\{w^i\} \in \mathcal{W}$, is defined by P^* such that $Z(P^*) = 0$. Under the assumptions mentioned above, we know that an equilibrium exists and the set $\mathcal{E} = \{P \in \Re^n_{++} : Z(P) = 0$ for some $\{w^i\} \in \mathcal{W}\}$ is non-empty.

[5] We shall use the superscript T to denote matrix transposition.

[6] $\|x\|$ stands for $\sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$, when $x = (x_1, x_2, \ldots x_n)$.

[7] One of the earliest in this connection was Arrow and Hahn (1971), Assumption 1, p. 293.

3.1 The Tatonnement Process

Consider an allocation $w^i \in \mathcal{W}$; unless otherwise stated this allocation will be held fixed in this section. A price adjustment process which one may consider is the following:

$$p_J = \gamma_j Z_j \; P \;\; for\; all\; j, \gamma_j > 0 \qquad (2)$$

This is what Negishi called the `non-normalized' system where the adjustment occurs on **all** prices.

A related process involves the choice of one good as the numeraire or the unit of account so that all prices are measured relative to good n; then the price vector is $P = (p, 1)$ and we may write $Z(P) = Z(p, 1) \equiv Z(p)$. The adjustment is then considered only on the relative prices p:

$$p_J = \gamma_j Z_j \; P \quad for\; all\; j \neq n,\; \gamma_j > 0 \qquad (3)$$

This process is called the `normalized' system. We shall consider mostly this system of equations.

Given our assumptions of the last section, for any initial price $P^0 = \; p^0, 1$, there is a solution to (3) denoted by $\emptyset_t \; p^0 \; = p(t)$, say. The equilibrium for the process[8] is $P^* = (p^*, 1)$ such that $Z(P^*) = Z(p^*, 1) \equiv Z(p^*) = 0$ and hence coincides with equilibrium for the economy, i.e., $P^* \in \Psi \quad w^i \quad \in \mathcal{E}$, where $\Psi: \mathcal{W} \to \mathcal{E}$ and thus associates with each distribution $w^i \in \mathcal{W}$ an equilibrium price configuration $P^* \in \mathcal{E}$. Given that the choice of the numeraire remains fixed, we shall refer to p^* as the equilibrium for the economy when $P^* = (p^*, 1) \in \mathcal{E}$; we shall in such cases, refer to $p^* \in \mathcal{E}$, where $\mathcal{E} = \; p: (p, 1) \in \mathcal{E}$. Consequently, we need to investigate whether $\emptyset_t \; p^0 \; \to p^* \in \mathcal{E}\; as\; t \to +\infty$. The stability of competitive equilibrium examines this question.

We shall say that the equilibrium p^* is globally stable under (3), if the solution $\emptyset_t \; p^0 \to p^*\; as\; t \to \infty$. for any arbitrary p^0; if convergence is ensured only under the condition that $p^0 \in N(p^*)$, where $N(p^*)$ is some neighborhood of p^*, then we shall say that p^* is locally stable under (3).

We mention the following two results in connection with the process (3)[9]:

1 Given the assumptions stated above, there exist $\varepsilon_i > 0, \forall\, i \neq n$ such that the solution to (3) $\emptyset_t \; p^0$ from any p^0 with $p_i^0 > \varepsilon_i, \forall\, i \neq n$ satisfies $p_i(t) > \varepsilon_i$ for all t > 0, $\forall\, i \neq n$.

In addition, we have:

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[8] That is where $p = 0$.

[9] For proofs, see Mukherji (2007)

2 The solution to (3), $\emptyset_t\ p^0$, from any p^0 with $p_i^0 > \varepsilon_i$ remains within a bounded subset of $\Re_{++}^n$.

In the above circumstances, we are assured that the solution to (3) has limit points within the positive orthant, provided that the initial price was strictly positive. Why study such processes? We have analyzed this question in some detail in Mukherji (2008) and (2010). We showed that if the endowments are redistributed appropriately, then a process like (2) is the modified gradient process for attaining an optimum for the economy; moreover, this process always converges. Thus price adjusting proportionally to the level of excess demand has some defense but only under the assumption that the distribution of endowments is proper. And when it is defensible, it works; that is, the solution converges. The investigation into Gale examples was the first indicator that the distribution of endowments has an important role to play. The gradient process and its properties is the second hint that we should be considering the role of the distribution of endowments. In Mukherji (2008), we had presented a regularity condition on the distribution of endowments which implied global stability of equilibrium. We shall present in the next section an extension of those results.

3.2 Sufficient Condition for Stability of Equilibrium

It should be noted then that the excess demand functions not only depend on the price p but also on the distribution of endowments w^i and we shall assume that

6. $Z_j\ p,\ w^i$, $Z_{jk}(p,\ w^i)$ for each j, k and any $p > 0$ are continuous in $w^i \in \mathcal{W}$.

Consider the matrix, the Jacobian of the excess demand functions defined as below:

$$J(p,\ w^i) = \begin{matrix} Z_{11} & \cdots & Z_{1n} \\ \ldots & \ldots & \ldots \\ Z_{n1} & \cdots & Z_{nn} \end{matrix}$$

where all the partial derivatives are evaluated at $p,\ w^i$. By using the properties introduced above, we have the following:

3 For any configuration $p,\ w^i$, $p > 0,\ w^i \in \mathcal{W}$, $we\ have, writing\ P = (p;\ 1)$

(a) $J\ p,\ w^i\ .P = 0$;

(b) $P^T.J\ p,\ w^i = -Z^T(p,\ w^i)$; and hence,

(c) $P^T.J\ p,\ w^i = 0\ if\ P \in \Psi\ \ w^i \subset \mathcal{E}$

The first is the homogeneity of degree zero in the prices; the second follows from differentiating the expression for Walras Law; and the last one follows from the second, using the definition of an equilibrium. It is clear therefore that the matrix $J(p, w^i)$ is singular at any configuration (p, w^i) and the matrix $J(p, w^i) + J^T(p, w^i)$ is singular if $P \in \Psi(w^i) \subset \mathcal{E}$; it may or may not be so elsewhere (i.e., out of equilibrium). Our final requirement may now be stated:

7. For any $w^i \in \mathcal{W}$, $J(p, w^i) + J^T(p, w^i)$ has rank $n-1$ whenever $P = (p, 1) \in \Psi(w^i) \subset \mathcal{E}$

Define $\mathcal{P} = [w^i \in \mathcal{W} :\sim \exists\ w^i \in \mathcal{W}$ such that $U^i(w^i) \geq U^i(w^i)\ \forall i$ with strict inequality for at least one i]: the set of *Pareto Optimal allocations*. We have the following:

4 $x^T.(J(p, w^i) + J^T(p, w^i)).x \leq 0\ \forall x \neq 0$, with the inequality strict if $x \neq \alpha P$ whenever $(p, 1) = P \in \Psi(w^i) \subset \mathcal{E}\ and\ w^i \in \mathcal{P}$.

Proof: The proof will be in two stages. The first part involves showing that at a Pareto Optimal allocation w^i, if $P \in \Psi(w^i) \subset \mathcal{E}$ then we have $P^T.Z(P, w^i) > 0 \forall P \neq \alpha P$; consequently the expression $f(P) = P^T.Z(P, w^i)$ attains a minimum at $P = P$ and hence, at $P = P$, the hessian matrix of the function $f(P), \nabla^2 f(P)$ must be positive semi-definite. Some tedious calculations establish that $-\nabla^2 f(P) = (J(p, w^i) + J^T(p, w^i))$ where $P = (p, 1)$. The claim then follows by invoking Assumption 7. In fact, the first part follows directly from an Arrow and Hurwicz Theorem (1958) which shows that if the distribution of endowments w^i is Pareto Optimal, then the Weak Axiom of Revealed Preference holds i.e., $P^T.Z(P, w^i) > 0 \forall P \neq \alpha P$ where $P \in \Psi(w^i) \subset \mathcal{E}$. So we have that the function $f(P)$ defined above attains a minimum at $P = P$. Now we observe, using Claim 3,(b), that

$$f_k(P) = \sum_j P_j Z_{jk}(P, w^i) = -Z_k(P, w^i) \Rightarrow f_k(P) = 0;$$

further we note, again using Claim 3 (c), that

$f_{kr}(p) = \sum_j P_j Z_{jkr}(p, w^i) \Rightarrow f_{kr}(p) = -[Z_{rk}(p, w^i) + Z_{kr}(p, w^i)]$ so that $-\nabla^2 f(P) = (J(p, w^i) + J^T(p, w^i))$ and hence positive semi definiteness of $-\nabla^2 f(P)$ implies that $(J(p, w^i) + J^T(p, w^i))$ is negative semi-definite; the matrix has rank $n-1$ by virtue of Assumption 7 and we know that $(J(p, w^i) + J^T(p, w^i)).P = 0$; so since

$x^T.(J(p, w^i) + J^T(p, w^i)).x \leq 0\ \forall x \neq 0$ and equality implies that

$(J(p, w^i) + J^T(p, w^i)).x = 0;$

the rank condition implies that $x = \alpha P$ and the claim follows. ■

We show next that it is possible to drop the requirement that w^i is Pareto Optimal and still deduce the above claim. In other words,

5 $x^T.\ (J(p,(w^i)) + J^T(p,(w^i))).x \leq 0\ \forall x \neq 0$, with the inequality strict if $x \neq \alpha P$ whenever $(p,1) = P \in \Psi((w^i)) \subset \mathcal{E}\ for\ any\ (w^i) \in \mathcal{W}, w^i > 0, P > 0.$

Proof: Suppose to the contrary that for some $(w^i) \in \mathcal{W}$, $P = (p,1) \in \Psi((w^i)) \subset \mathcal{E}$ we have the matrix $J(p,(w^i)) + J^T(p,(w^i))$ is not negative semi-definite; i.e., it has at least one positive characteristic root.

Let y^i solve for each i the following maximum problem:

$\max_y U^i(y)$ subject to $P^T.y \leq P^T.w^i$

Then y^i is the **demand** by i at the equilibrium P; and $(y^i) \in \mathcal{P}$, a Pareto Optimal allocation. Note that for any (w^i_α), where $w^i_\alpha = \alpha w^i + (1-\alpha)y^i, 0 \leq \alpha \leq 1$, since $P^T.w^i_\alpha = P^T.w^i\ \forall\ \alpha \in [0,1]$, demands at prices P remain unaltered and hence $P \in \Psi((w^i_\alpha))$ for any value of $\alpha \in [0,1]$.

Note that $J(p,(y^i)) + J^T(p,(y^i))$ is negative semi-definite with rank $n-1$ i.e.,there are $n-1$ negative characteristic roots and a single zero characteristic root.

Define

$$\alpha = \sup_{\alpha \in [0,1]}\{\alpha : J(p,(w^i_\alpha)) + J^T(p,(w^i_\alpha)) \text{ has } n-1 \text{ negative roots}\}$$

The supremum exists since by assumption for $\alpha = 1$, the relevant matrix has a positive root and hence has less than $n-1$ negative roots; for $\alpha = 0$ there are $n-1$ negative roots. Thus the set is non-empty since 0 belongs to the set and bounded above < 1. It is clear that for $\alpha = \alpha$ the matrix $J(p,(w^i_\alpha)) + J^T(p,(w^i_\alpha))$ has $n-2$ negative roots with 0 as a repeated root; since otherwise, a slightly larger value for α would also be eligible. But this means that $J(p,(w^i_\alpha)) + J^T(p,(w^i_\alpha))$ has rank $n-2$ at $\alpha = \alpha$ [10]: this contradicts Assumption 7 since $P = (p,1) \in \Psi((w^i_\alpha))$, as we discussed above. Hence there can be no such $(w^i) \in \mathcal{W}$, $P = (p,1) \in \Psi((w^i)) \subset \mathcal{E}$. ■

Thus note that the above means that

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[10] This deduction may be made only because the relevant matrix is symmetric.

6 For any $w^i \in \mathcal{W}$ and any $P \in \Psi \quad w^i \subset \mathcal{E}$, we must have $J\ p\ ,\ w^i \ + J^T\ p\ ,\ w^i$ negative semi-definite with rank $n - 1$. Thus given any $w^i \in \mathcal{W}$, $P = \ p, 1 \ \in \Psi \quad w^i$ is locally asymptotically stable under a process such as (3) and hence for every $w^i \in \mathcal{W}$ there is a unique equilibrium $P = \ p, 1 \ \in \Psi \quad w^i$.

Proof: We observe that given some $w^i \in \mathcal{W}$, $P = \ p, 1 \ \in \Psi \quad w^i \subset \mathcal{E}$, we have shown that $J\ p\ ,\ w^i \ + J^T\ p\ ,\ w^i$ negative semi-definite with rank $n - 1$; since w^i will remain fixed we shall drop this from the arguments of the matrices and simplify notation further by writing $J\ p \ + J^T\ p \ = B\ p$, say. Now to verify local asymptotic stability of the equilibrium P under the process (3), we linearize this process around the equilibrium and we get

$$x = \Lambda.\ J(p)x \tag{4}$$

where $x \in \Re^{n-1}$, $x = p - p$, Λ is a diagonal matrix of order $(n - 1)$ with λ_j in the $jj - th$ entry. Further $J(p)$ is the first $(n - 1)$ rows and columns of $J(p)$. Notice that $J\ p \ + J^T\ p$ must be negative semi-definite being a principal minor of $J\ p \ + J^T\ p$ and in fact must be negative definite, since otherwise rank of $J\ p \ + J^T\ p \ \leq (n-2)$: a contradiction to Assumption 7. Now consider $V(t) = \ _j x_j^2(t)/\lambda_j$ where $x(t)$ is the solution to (4). Note that $V\ t \ = 2x\ t\ ^T.\ J\ p\ .x\ t \ = x\ t\ ^T.\ \ J\ p \ + J^T\ p \ .x(t) < 0$ unless $x(t) = 0$; this allows us to conclude that $x(t) \to 0$ and hence that the equilibrium is locally asymptotically stable. Since this is so for every equilibrium p such that $p, 1 \ \in \Psi \quad w^i$, one may use a theorem of Arrow and Hahn (1971) to conclude that $\Psi \quad w^i$ is a function and that the equilibrium is unique given w^i. ■

Finally note that we have on the basis of our assumptions shown that there is a unique equilibrium which is locally asymptotically stable under the process (3). There is another point which needs to be noted and this relates to the situation when the condition 7 is violated. Notice now that unstable positions of equilibrium are possible. In particular suppose that at some $w^i \in \mathcal{W}$, $P \in \Psi \quad w^i \subset \mathcal{E}$, P is unstable i.e., the matrix $J\ p \ + J^T\ p$ has a characteristic root which is non-negative. (As for example in the Gale example, this was positive). If the demands at this equilibrium are given by the array y^i and if the rank of $J\ p\ ,\ y^i \ + J^T\ p\ ,\ y^i$ is full (i.e., $n - 1$) then a redistribution of the endowments, as in the case of the Gale example, will lead to a stable equilibrium; and one need not eliminate all trade to arrive at a stable equilibrium.

Consider then the considerable weakening of assumption 7:

8. $J\ \ p,\ w^i\ \ \ + J^T\ \ p,\ w^i$ has rank $n-1$ whenever $P \in \Psi\ \ \ w^i\ \ \ \subset \mathcal{E}$ and $w^i\ \ \in \mathcal{P}$.

We may state the immediate preceding discussion in the form of the following:

7 Under assumption 8, if for any $w^i\ \ \in \mathcal{W}$ the associated equilibrium P is unstable, then there is a redistribution of the endowments $w^i\ \ \in \mathcal{W}$ which would maintain the same P as equilibrium and for which P is locally asymptotically stable and there is some trade at the equilibrium prices.

In the above, the condition 7 has been weakened considerably: now we require that this be satisfied only at zero trade equilibria. The proof follows since if at the original distribution of endowments, y^i denotes the array of demand at the equilibrium P, we know that $J\ \ p,\ y^i\ \ \ + J^T\ \ p,\ y^i$ is negative semi-definite and hence there would be some redistribution lying on the convex combination of y^i and w^i which yields the desired outcome and such redistributions need not necessarily be the demand array. The above claim qualifies to be called the Second Fundamental Theorem of Positive Economics.

Remark 1 *On conditions 7 and 8. The former, as we saw, was a strong assumption which among other results, implied the uniqueness of competitive equilibrium. However a weaker assumption may not imply the stability result. Consider, for example the Scarf example. One may compute the matrix* $J\ \ p\ \ + J^T\ \ p$ matrix and show that it is the null matrix; 7 is violated[11].

For the Gale example, re- call our analysis; the function $f(p, 2/3) = 0 \forall p$ implies that when the endowments are $(2/3, 1/3)$ for A and $(1/3, 2/3)$ for B, the excess demand curve is horizontal and hence every price is an equilibrium: the crucial assumption 7 is violated once again. The assumption 8 on the other hand, appears to be considerably weaker since it is a requirement for only those distributions which are Pareto optimal. In fact one might expect this property to be

[11] See, for example, Mukherji (2007)

satisfied for generic economies[12]. Once this condition holds, then it is always possible to deduce stability of equilibrium by properly redistributing endowments.

4 Related Literature and Implications of the above exercises

We begin by noting that there are several strands of literature related to our investigation; first the stability of competitive equilibrium literature. Our results may be seen as a first step in the direction outlined at the beginning. This aspect led to what we have called the Second Fundamental Theorem of Positive Economics, (SFTPE) for reasons which should be apparent.
It should be noted that studies relating the distribution of endowments to stability of equilibrium exist, e.g., Hirota (1981) and (1985); however the result obtained above is different from these results. For specific preference patterns, though a weaker version of our result may be seen to follow from the Hirota studies.

We should also point out that there have been some related studies which try to investigate the results that may be obtained by aggregating across individuals. Two such works are due to Hildenbrand (1983) and Grandmont (1992). The result of the former, market demand having a quasi- negative definite Jacobian (identical to $J + J^T$ being negative definite with rank $n - 1$) is is obtained by aggregation **only if** endowments are collinear; it should be noted that the assumption is much stronger than the ones we have employed. The second later study considers agents' characteristics in terms of a pair: preferences and income; the starting point of this analysis is a transformation indexed by $\alpha = (\alpha_1, \_\_\_, \alpha_n)$[13] of the commodity space. Consequently agents characteristics are expressed in terms of a marginal distribution over the space of preferences and income and for each preference and income, a conditional distribution over all transforms α. If every commodity is desired in the aggregate (a version of our boundary assumption) and if the conditional distribution over all transforms, given a preference and income, has a density which is fat enough then aggregate demand has very nice properties as for example gross substitution on a set of prices whose size is shown to depend on the degree of behavioral heterogeneity (the density being fatter implies increased heterogeneity).

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[12] I am indebted to the referee for this remark.

[13] The axis corresponding to good j is stretched by e^{α_j}.

There is another strand of literature to which our analysis may be seen to be related. This concerns the search for adjustment processes which converge to equilibrium; these adjustment processes may be difficult to pin down as ones which reflect the behavior of prices in disequilibrium but nevertheless their convergence properties are somewhat superior. The literature began with Smale (1967) and consists of contributions from van der Laan and Talman (1987), Kamiya (1990), Mukherji (1995) and Herings (1997).

The approach here has been to analyze the properties of adjustment processes which yield better convergence results. In contrast, in the current paper we return to the usual price adjustment process and look at redistribution of endowments to yield better convergence. Our results would also have a bearing on the class of results which have been referred to in the literature as "Anything Goes" theorems or the Sonnenschein-Debreu-Mantel (1972-74) theorems. Basically the claim there was that the properties of Walras Law and homogeneity of degree zero in the prices do not restrict excess demand functions in any significant way. More importantly, given any set of functions which satisfy Walras Law and homogeneity of degree zero in the concerned variables may be obtained as excess demand functions for an appropriate economy. Consequently neither stability nor uniqueness could be assumed from maximizing behavior. While this result has been taken to be a robust conclusion, and hence largely negative, our result shows that all is not lost since by redistributing the initial resources we may obtain the desired features of equilibrium.

Thus the main result claimed as the Second Fundamental Theorem of Positive Economics (SFTPE) appears to be based on weak assumptions and to reiterate, **states that for any exchange economy, for any set of preferences, any competitive equilibrium price configuration given some distribution of endowments, may be made locally asymptotically stable by redistributing the endowments in such a way that the equilibrium price is unaltered, and trades take place with the new distribution at the equilibrium**. The only restriction required is that the Jacobian $J + J^T$ at zero trade equilibrium have maximal rank (i.e., $n - 1$). The Second Fundamental Theorem of Welfare Economics also requires some restrictions as readers may recall: convexity of preferences and the assumption that Professor McKenzie taught his students to refer to as the existence of the `cheaper point'.

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