Monetary and Fiscal Policy in the Presence of Informal Labour Markets*

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Abstract

How does informality in emerging economies affect the conduct of monetary and fiscal policy? To answer this question we construct a two-sector, formal-informal new Keynesian closed-economy. The informal sector is more labour intensive, is untaxed, has a classical labour market, faces high credit constraints in financing investment and is less visible in terms of observed output. We compare outcomes under welfare-optimal monetary policy, discretion and welfare-optimized interest-rate Taylor rules alongside a balanced-budget fiscal regime. We compare the model, first with no frictions in these two markets, then with frictions in only the formal labour market and finally with frictions on both credit markets and the formal labour market. Our main conclusions are first, labour and financial market frictions, the latter assumed to be stronger in the informal sector, cause the time-inconsistency problem to worsen. The importance of commitment therefore increases in economies characterized by a large informal sector with the features we have highlighted. Simple implementable optimized rules that respond only to observed aggregate inflation and formal-sector output can be significantly worse in welfare terms than their optimal counterpart, but are still far better than discretion. Simple rules that respond, if possible, to the risk premium in the formal sector result in a significant welfare improvement.

JEL Classification: J65, E24, E26, E32

Keywords: Informal economy, emerging economies, labour market, credit market, tax policy, interest rate rules

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1 Introduction

Relatively little has been written on the conduct of monetary policies in economies with a large informal sector. This is a reality in many emerging and in some OECD countries and the study of informality can shed new lights on the impact of labour markets and monetary policies on the economic performance of these economies.

The phenomenon, that we refer in our paper as 'informality' has been discussed using different terminology: unregistered, hidden, shadow, unofficial underground and, in a more restrictive sense black, economy. The term 'hidden economy' has often been used with respect to advanced economies, whilst the term 'informal economy' has been usually been applied to developing economies. Chen (2007) describes the move from the 'old' concept of the informal sector to a more comprehensive view of the informal economy. The 'new' view of informality which focuses on the worker and informal employment, that is employment without any sort of protection, includes self-employment in unregistered firms and wage employment in unprotected jobs. According to the definition used, the estimates of the size of the informal economy can be very different.[1]

In general, different modelling strategies apply to describe different phenomena. It is possible to distinguish informality in the goods market, informality in the credit market and informality in the labour market.[2] Here we focus on the latter two phenonema, and study the impact of monetary policy in an economy where the size of the informal labour market is driven by the taxation regime and the differing frictions in the two sectors. This model describes an economy with two sectors producing two different goods. In equilibrium, workers who do not find a job in the unionized formal labour market (i.e. the sector with a higher labour standard), move to the informal sector. In our model public goods are produced formally and the two sectors have different technologies, the informal sector being more labour intensive. Further distinctions are the introduction of a market friction in the labour market in the formal sector, whilst the informal sector is frictionless in this respect and the presence of financial accelerators of differing impact in the two

[1] Informality is also defined in different ways in various countries. For example, in India, the informal sector is generally identified with the unorganized sector (no legal provision and no regular accounts). According to the NCEUS report on Definitional and Statistical Issues relating to the Informal Economy, the informal (or unorganized) economy is given by the informal (or unorganized) sector and its workers plus the informal workers in the formal sector, where the unorganized (informal) sector is defined as all incorporated private enterprises owned by individuals/households with less than 10 workers. Also the report defines unorganized (informal) workers, as workers in the unorganized sector, households, excluding regular workers with social security benefits, plus workers in the formal sector without any social security benefit NCEUS (2008).

[2] See Batini *et al.* (2010) for details.

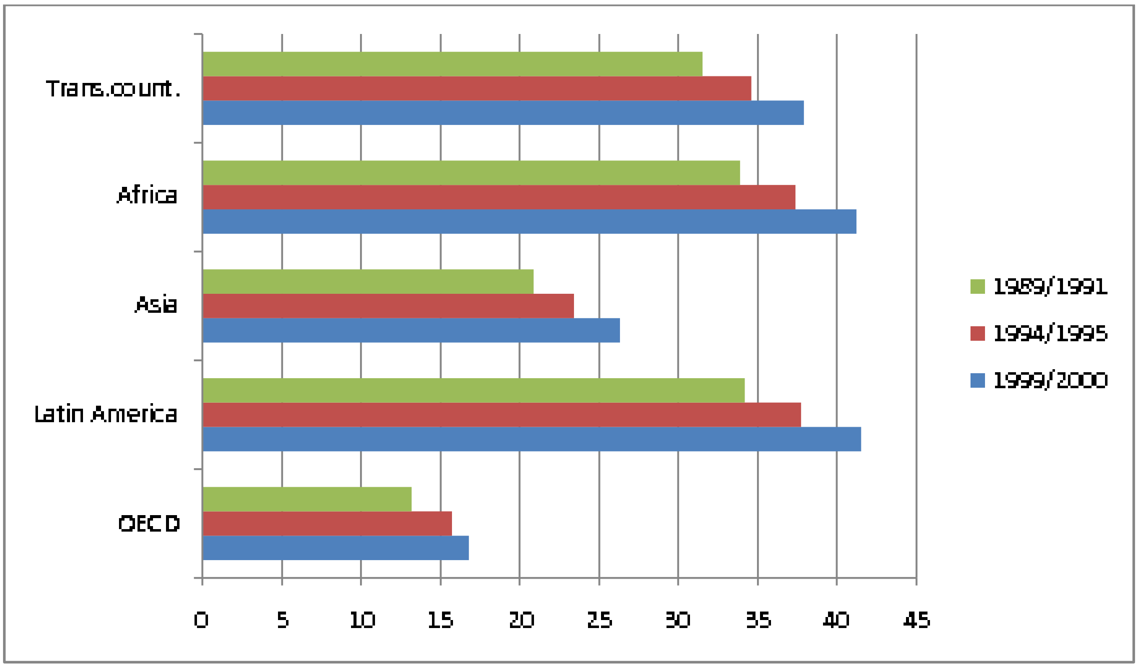

Figure 1: **Size of Informal Economy around the World Schneider (2005)**

sectors. Thus we capture some of the main characteristics of the informal sector: labour-intensiveness, lack of public goods production and wage flexibility and credit constraints. We also consider, in relation to the conduct of monetary policy a further dimension of informality: its lack of transparency. Price stickiness is added to both sector, but with the informal sector relatively flexible to give us the New Keynesian aspect and a model that can be used to investigate the flows between formal and informal sector and the link between inflation and unemployment, i.e. the Phillips curves in countries with a large informal economy.

Table 1: **Characterizing Informality**

	Labour Market	**Credit Market**	**Taxation**	**Visibility**	**Price Stickiness**
F Sector	frictions	lower frictions	taxed	higher	high
I Sector	no frictions	higher frictions	untaxed	lower	low

We study optimal monetary policy and consider the extent to which the difficulty in observing the informal sector affects its efficacy.

The modelling approach also captures the *a priori* ambiguous impact of informality. On the one hand, the flexible and frictionless informal labour market reduces business cycle costs. On the other hand, informality brings various costs due to the realistic assumption that it lies outside the tax regime and it is often unobservable.

The remainder of the paper is organized as follows. Section 2 shows how our general equilibrium economy relates to similar theoretical frameworks within the DSGE and the informal economy literatures. Section 3 sets out details of our model. Section 4 analyzes steady-state properties and the calibration based on the steady state. Section 5 studies optimal monetary policy with and without commitment. Section 6 examines simple Taylor-type commitment rules and their ability to mimic their optimal counterpart. Section 7 concludes the paper.

2 Background Literature

Satchi and Temple (2009) and Marjit and Kar (2008) recognize the importance of a *general equilibrium analysis*, but they often make assumptions which usually do not allow for a comprehensive view of causes and effects of informality. For example, standard assumptions in the search-matching literature (i.e. linear utility function among the other), exclude the consumption-hours decision. Conesa *et al.* (2002) represents an attempt in this direction. The author describes a simple RBC model with an informal sector. They introduce a second sector into a standard Real Business Cycle (RBC) model which is described as an "underground" economy that has a different technology, produces goods and services that could otherwise be produced in the formal sector, but is not registered in NI accounts. The main characteristics of the model includes: a wage premium which can be seen as the opportunity cost of not working in the official sector and labour indivisibilities in the formal/registered sector. Households choose a probability of working in the informal sector which can be interpreted as the purchase of lotteries in a perfectly insured market. When a worker chooses the informal sector he/she enjoy more leisure at the price of a smaller wage, while in the formal sector individuals work more, but receive a wage premium. In particular, the authors assume labour to be indivisible in the formal/registered sector with hours worked fixed exogenously. The main prediction of the model is that wage premium differentials can explain the different size of macroeconomic fluctuations in function of technological shocks. The intuition is the following: countries with a smaller wage premium have a lower opportunity cost to participate in the formal sector and so they have smaller participation rates. In those countries, the effects of technological shocks are amplified.

A series of papers incorporate the search and matching approach into DSGE models to explain the cyclical behaviour of employment, job creation, job destruction and inflation rate in response to a monetary policy shock, but there are several unexplored area left for research. [3] In general there is a rapidly growing literature on search and match labour market in New Keynesian DSGE models in addition to Ravenna and Walsh (2007). Christiano *et al.* (2007), Sala *et al.* (2008) and Thomas (2008) introduce labour market frictions in New Keynesian models allowing the study of, both, the intensive and the extensive margin of labour usage during the business cycle. Blanchard and Gali (2007) adopt a simpler hiring cost approach in a New Keynesian framework.

Castillo and Montoro (2008) develop Blanchard and Gali (2007) by modelling a labour market economy with formal and informal labour contracts within a New Keynesian model with labour market frictions. This is the first paper that analyses together the creation of informal jobs and the interaction between the informal sector and monetary policy. Informality is a result of hiring costs, which are a function of the labour market tightness. In equilibrium , firms in the wholesale sectors balance the higher productivity of a formal production process with the lower hiring costs of the informal process. Marginal costs will then become a function also of the proportion of informal jobs in the economy. The interesting results of this theoretical framework is that during period of high aggregate demand the informal sector expands due to lower hiring costs associated with this technology. This creates a link between informality, the dynamics of inflation and the transmission mechanism of monetary policy. In particular, the authors show that *"informal workers act as a buffer stock of labour that allows firms to expand output without putting pressure on wages"*. Castillo and Montoro (2008) allow for a voluntary decision where the marginal worker is indifferent between formal and informal sector. Labour market regulations may reduce labour demand without introducing segmentation per se. While we recognize this picture is realistic in many advanced economies and there is also evidence that shows the existence of a voluntary, small firms sector in some developing countries as discussed in Perry *et al.* (2007), we believe that in the majority of the developing world informality is a result of *segmentation* where workers turn to the informal labour market when they cannot find a job in the formal sector. As in Satchi and Temple (2009) and Marjit and Kar (2008) we model the idea that: *"Unemployment is a luxury"* and that *"informal sector activities provide an unofficial safety-net in the absence of state-provided unemployment insurance"*.

As in Zenou (2008), we allow for a frictionless informal labour market. We also in-

[3] See Yasgiv (2007) for a survey on the developments of search-matching models and (Ravenna and Walsh (2007)) for a recent application of search frictions in New Keynesiam models.

troduce labour market frictions in the formal sector, but we do not explicitly model the matching process as in these papers. Rather we follow another modelling option favoured in the literature by introducing a wage norm in the formal sector. While we explore the general equilibrium features of informality, our model is in line with the Harris and Todaro tradition (Harris and Todaro, 1970) in describing a very simple labour market structure where labour in the formal sector is fixed at a higher than the market clearing level. See also Marjit and Kar (2008) and Agenor and Montiel (1996) for a similar assumption. As discussed in Satchi and Temple (2009) a richer labour market structure implies a wage in the formal sector which is endogenously determined. While this can be a promising future development we believe the simplifying assumption allows us to obtain interesting conclusions without adding further complications to the already complex modelling framework. In this respect, we should also mention that, following the critics on the inability of the search matching model to generate the observed unemployment volatility as reported in Shimer (2005), a series of papers depart from the flexible wage assumption in order to generate enough volatility in the unemployment rate (see Blanchard and Gali (2007), Krause and Lubik (2007) and Christoffel and Linzert (2005)). The introduction of a real wage norm in New Keynesian models has been described as one of the possible way to reconcile the model with the data. [4]

As clarified in Zenou's paper *".. in the informal sector, either people are self-employed or work with relatives or friends and thus do not apply formally for jobs posted in newspapers or employment agency"*. As in Zenou (2008) we do not model this idea explicitly, but our competitive informal labour market implies free-entry and an instantaneous hiring process. Zenou's framework has no NK features and focuses on the evaluation of various labour market policies on the unemployment rate of an economy with an informal sector.

Our paper contribution to this literature is as follows. We look at the efficacy of monetary policy and for this reason we require a more general framework where households' consumption and leisure decisions are explicitly modelled in a DSGE model with price rigidity. We introduce New Keynesian price rigidities in the usual way, as in Castillo and Montoro (2008), but then proceed to analyse the interaction of informal and formal sectors and the implications for monetary policy. Our analysis of simple optimized Taylor-type interest rate rules, the incorporation of zero-lower bound constraint and the comparison between simple and optimal rules where one sector is unobserved are particularly novel

[4] However the introduction of such real wage rigidity is not immune of critics. Thomas (2008) introduces staggered nominal wages and points to a series of advantages of his approach with respect to the real wage norm assumption while Hornstein *et al.* (2005) and Pissarides (2008) claim that wage rigidity needs to be accompanied by an unrealistic assumption on the labour share and points instead at the introduction of demand shocks as a possible solution to the unemployment volatility puzzle.

features for the informal economy literature.

3 The Model

Consider a two-sector "Formal" (F) and "Informal" (I) economy, producing different goods with different technologies which sell at different retail prices, $P_{F,t}$ and $P_{I,t}$, say. Labour and capital are the variable factor inputs and the informal sector is less capital intensive. In both F and I sectors competitive wholesale firms supply a homogeneous good (in that sector) to a monopolistic retail sector. The latter convert the homogeneous good in a differentiated product that is sold above its marginal cost. In a free-entry equilibrium, monopolistic profits are driven down to the conversion costs so all firms, wholesale or retail, F or I, make zero profits and are indifferent as to which sector they operate.

Government services are provided from the formal sector and is financed by an employment tax as in Zenou (2008). In the general set-up this can be shared by the formal and informal sectors giving us a framework in which the role of tax incidence can be studied as one of the drivers of informalization. The other drivers in our model are the degree of real wage rigidity in the formal sector and the credit constraints facing the informal sector. But first we set out a model without these latter two frictions in the labour and financial markets.

3.1 The Model without Labour and Credit Market Frictions

3.1.1 Households

A proportion $n_{F,t}$ of household members work in the formal sector. Hours $h_{F,t}$ and hours $h_{I,t}$ are supplied in the F and I sectors respectively. Members who work in sector $i = I, F$ derive utility $U(C_t, L_{i,t})$ where C_t is household shared consumption and leisure $L_{i,t} = 1 - h_{i,t}$ and we assume that[5]

$$U_C > 0,\ U_L > 0,\ U_{CC} \leq 0,\ U_{LL} \leq 0 \tag{1}$$

. The representative household single-period utility is

$$\Lambda_t = \Lambda(C_t, n_{F,t}, h_{F,t}, h_{I,t}) = n_{F,t} U(C_t, 1 - h_{F,t}) + (1 - n_{F,t}) U(C_t, 1 - h_{I,t}) \tag{2}$$

We construct Dixit-Stiglitz consumption and price aggregates

$$C_t = \left[\mathrm{w}^{\frac{1}{\mu}} C_{F,t}^{\frac{\mu-1}{\mu}} + (1-\mathrm{w})^{\frac{1}{\mu}} C_{I,t}^{\frac{\mu-1}{\mu}} \right]^{\frac{\mu}{\mu-1}} \tag{3}$$

$$P_t = \left[\mathrm{w}(P_{F,t})^{1-\mu} + (1-\mathrm{w})(P_{I,t})^{1-\mu} \right]^{\frac{1}{1-\mu}} \tag{4}$$

[5]Our notation is $U_C \equiv \frac{\partial U}{\partial C}$, $U_L \equiv \frac{\partial U(C,L)}{\partial L}$, $U_{CC} \equiv \frac{\partial^2 U}{\partial C^2}$ etc.

Then standard inter-temporal and intra-temporal decisions lead to

$$\frac{\Lambda_{C,t}}{P_t} = \beta E_t \left[(1+R_{n,t}) \frac{\Lambda_{C,t+1}}{P_{t+1}} \right] \tag{5}$$

$$C_{F,t} = \mathrm{w} \left(\frac{P_{F,t}}{P_t} \right)^{-\mu} C_t \tag{6}$$

$$C_{I,t} = (1-\mathrm{w}) \left(\frac{P_{I,t}}{P_t} \right)^{-\mu} C_t \tag{7}$$

where $R_{n,t}$ is the nominal interest rate over the interval $[t, t+1]$ for riskless bonds set by the central bank at the beginning of the period. Note that substituting (6) and (7) into (3) gives (4), so that (4) or (3) are superfluous for the set-up. Total labour supply is found by equating the marginal rate of substitution between labour and leisure with the real wages for the two sectors:

$$\frac{U_{L_I,t}}{\Lambda_{C,t}} = \frac{W_{I,t}}{P_t} \tag{8}$$

$$\frac{U_{L_F,t}}{\Lambda_{C,t}} = \frac{W_{F,t}}{P_t} \tag{9}$$

For the moment there are no labour market frictions so we have $W_{I,t} = W_{F,t} = W_t$, say. It follows from (8) that $L_{I,t} = L_{F,t}$ and $h_{I,t} = h_{F,t} = h_t$, say.

3.1.2 Capital Producing Firms

It is convenient to introduce capital producing firms that at time t convert I_t of output into $(1 - S(X_t))I_t$ of new capital sold at a real price Q_t. The law of motion of capital is then given by

$$K_{t+1} = (1-\delta)K_t + (1 - S(X_t))I_t\,;\;\; S',\, S'' \geq 0\,;\;\; S(1+g) = S'(1+g) = 0 \tag{10}$$

$$X_t \equiv \frac{I_t}{I_{t-1}} \tag{11}$$

Capital producing firms then maximize expected discounted profits

$$E_t \sum_{k=0}^{\infty} D_{t,t+k} \left[Q_{t+k}(1 - S\left(I_{t+k}/I_{t+k-1}\right))I_{t+k} - I_{t+k} \right]$$

where $D_{t,t+k}$ is the real stochastic discount rate. This results in the first-order condition

$$Q_t(1 - S(X_t) - X_t S'(X_t)) + E_t \left[D_{t,t+1}\, Q_{t+1} S'(X_{t+1}) \frac{I_{t+1}^2}{I_t^2} \right] = 1 \tag{12}$$

Up to a first order approx this is the same as

$$Q_t(1 - S(X_t) - X_t S'(X_t)) + E_t \left[\frac{1}{(1+R_{t+1})}\, Q_{t+1} S'(X_{t+1}) \frac{I_{t+1}^2}{I_t^2} \right] = 1 \tag{13}$$

3.1.3 Wholesale Firms

Wholesale output in the two sectors is given by a Cobb-Douglas production function

$$Y_{i,t}^W = F(A_{i,t}, N_{i,t}, K_{i,t}),\ i = I, F \tag{14}$$

where $A_{i,t}$ are a technology, total labour supply $N_{i,t} = n_{i,t}h_{i,t}$, $i = I, F$. Capital inputs are $K_{i,t}$, $i = I, F$ and we assume capital is accumulated from formal output only.

The first-order condition for labour demand are

$$P_{F,t}^W F_{N_F,t} = W_{F,t} + P_t\tau_{F,t} \tag{15}$$

$$P_{I,t}^W F_{N_I,t} = W_{I,t} + P_t\tau_{I,t} \tag{16}$$

$$\tag{17}$$

where $P_{F,t}^W$ and $P_{I,t}^W$ are wholesale prices, $\tau_{F,t}$, $\tau_{F,t}$ are the employment real tax rates in the formal sector and informal sectors respectively.

Demand for capital by firms must satisfy

$$\begin{aligned} E_t[1 + R_{t+1}] &= \frac{E_t\left[\frac{P_{F,t+1}^W}{P_t}F_{K_F,t+1} + (1-\delta)Q_{t+1}\right]}{Q_t} \equiv E_t[1 + R_{F,t+1}^k] \\ &= \frac{E_t\left[\frac{P_{I,t+1}^W}{P_t}F_{K_I,t+1} + (1-\delta)Q_{t+1}\right]}{Q_t} \equiv E_t[1 + R_{I,t+1}^k] \end{aligned} \tag{18}$$

In (18) the rhs is the expected gross return to holding a unit of capital in from t to $t+1$ in the two sectors prespectively. The lhs is the expected gross return from holding bonds, the opportunity cost of capital.

3.1.4 Retail Firms

We now introduce a retail sector of monopolistic firms within each sector buying wholesale goods and differentiating the product at a proportional resource cost $c_iY_{i,t}^W$ in sectors $i = F, I$. In a free-entry equilibrium profits are driven to zero. Retail output for firm f in sector is then $Y_{i,t}(f) = (1 - c_i)Y_{i,t}^W(f)$ where $Y_{i,t}^W$ is produced according to the production technology (14) at prices $P_{i,t}^W$. Let the number of differentiated varieties produced in the informal and formal sectors be ν_F and ν_I respectively. Each is produced by a single retail firm and the numbers of these firms is fixed.[6] Let $C_{F,t}(f)$ and $C_{I,t}(f)$ denote the home

[6] This model structure closely follows a model of two interacting economies in the New Open Economy Literature.

consumption of the representative household of variety f produced in sectors F and I. Aggregate consumption of each category now become indices

$$C_{F,t} = \left[\left(\frac{1}{\nu_F}\right)^{\frac{1}{\zeta_F}} \sum_{f=1}^{\nu_F} C_{F,t}(f)^{(\zeta_F-1)/\zeta_F}\right]^{\zeta_F/(\zeta_F-1)}$$

$$C_{I,t} = \left[\left(\frac{1}{\nu_I}\right)^{\frac{1}{\zeta_I}} \left(\sum_{f=1}^{\nu_I} C_{I,t}(f)^{(\zeta_I-1)/\zeta_I}\right)\right]^{\zeta_I/(\zeta_I-1)}$$

where $\zeta_F, \zeta_I > 1$ are the elasticities of substitution between varieties in the two sectors. Aggregate output is similarly defined:

$$Y_{F,t} = \left[\left(\frac{1}{\nu_F}\right)^{\frac{1}{\zeta_F}} \sum_{f=1}^{\nu_F} Y_{F,t}(f)^{(\zeta_F-1)/\zeta_F}\right]^{\zeta_F/(\zeta_F-1)}$$

$$Y_{I,t} = \left[\left(\frac{1}{\nu_I}\right)^{\frac{1}{\zeta_I}} \left(\sum_{f=1}^{\nu_I} Y_{I,t}(f)^{(\zeta_I-1)/\zeta_I}\right)\right]^{\zeta_I/(\zeta_I-1)}$$

Then the optimal intra-sectoral decisions are given by standard results:

$$C_{F,t}(f) = \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\zeta_F} C_{F,t} \tag{19}$$

$$C_{I,t}(f) = \left(\frac{P_{I,t}(f)}{P_{I,t}}\right)^{-\zeta_I} C_{I,t} \tag{20}$$

and inter-sector decisions are as before.

We introduce endogenous investment, I_t, and exogenous government spending G_t both assumed to consist entirely of formal output. Then maximizing the investment and government expenditure indices as for the consumer in (19) we have

$$I_t(f) = \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\zeta_F} I_t \tag{21}$$

$$G_t(f) = \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\zeta_F} G_t \tag{22}$$

Using (19)–(22) it follows that total demands for each differentiated product are given by

$$Y_{F,t}(f) = C_{F,t}(f) + I_t(f) + G_t(f) = \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\zeta_F} (C_{F,t} + I_t + G_t) = \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\zeta_F} Y_{F,t} \tag{23}$$

$$Y_{I,t}(f) = C_{I,t}(f) = \left(\frac{P_{I,t}(f)}{P_{I,t}}\right)^{-\zeta_I} C_{I,t} = \left(\frac{P_{I,t}(f)}{P_{I,t}}\right)^{-\zeta_I} Y_{I,t} \tag{24}$$

Retail firms follow Calvo pricing. In sector $i = F, I$, assume that there is a probability of $1 - \xi_i$ at each period that the price of each good f is set optimally to $\hat{P}_{i,t}(f)$. If the price is not re-optimized, then it is held constant.[7] For each producer f the objective is at time t to choose $\hat{P}_{i,t}(f)$ to maximize discounted profits

$$E_t \sum_{k=0}^{\infty} \xi_i^k D_{t,t+k} Y_{i,t+k}(f) \left[\hat{P}_{i,t}(f) - P_{i,t+k} \mathrm{MC}_{i,t+k} \right]$$

where $D_{t,t+k}$ is the discount factor over the interval $[t, t+k]$, subject to a downward sloping demand from consumers of elasticity ζ_i given by (23) and (24), and $MC_{i,t} = \frac{P^W_{i,t}}{P_{i,t}}$ are real marginal costs. The solution to this is

$$E_t \sum_{k=0}^{\infty} \xi_i^k D_{t,t+k} Y_{i,t+k}(f) \left[\hat{P}_{i,t}(f) - \frac{\zeta_i}{(\zeta_i - 1)} P_{i,t+k} \mathrm{MC}_{i,t+k} \right] = 0 \tag{25}$$

and by the law of large numbers the evolution of the price index is given by

$$P_{i,t+1}^{1-\zeta_i} = \xi_i \left(P_{i,t}\right)^{1-\zeta_i} + (1 - \xi_i)(\hat{P}_{i,t+1}(f))^{1-\zeta_i} \tag{26}$$

These summations can be expressed as difference equations as follows. First define for $i = I, F$, $\Pi_{i,t} \equiv \frac{P_{i,t}}{P_{i,t-1}} = \pi_{i,t} + 1$. Then from the Euler equation we have that $D_{t+k,t} = \beta^k \frac{U_{C,t+k}}{U_{C,t}}$. Using this result we can derive the aggregate price dynamics for $i = I, F$ as

$$\begin{aligned}
H_{i,t} - \xi_i \beta E_t[\Pi_{i,t+1}^{\zeta_i - 1} H_{i,t+1}] &= Y_{i,t} U_{C,t} \\
J_{i,t} - \xi_i \beta E_t[\Pi_{i,t+1}^{\zeta_i} J_{i,t+1}] &= \left(\frac{1}{1 - \frac{1}{\zeta_i}} \right) Y_{i,t} U_{C,t} MS_{i,t} MC_{i,t} \\
\frac{\hat{P}_{i,t}}{P_{i,t}} H_{i,t} &= J_{i,t} \\
1 &= \xi_i \Pi_t^{\zeta_i - 1} + (1 - \xi_i) \left(\frac{\hat{P}_{i,t}}{P_{i,t}} \right)^{1-\zeta_i}
\end{aligned}$$

where $MS_{i,t}$ is an exogenous mark-up shock in sector $i = I, F$.[8]

3.1.5 Equilibrium

Assuming Cobb-Douglas technology in the wholesale sectors (see all functional forms below) for each differentiated product in the F and I sectors we equate supply and demand

[7] Thus we can interpret $\frac{1}{1-\xi_i}$ as the average duration for which prices are left unchanged in sector $i = F, I$.

[8] These shocks are scaled in the calibration so that a one standard deviation impulse response leads to an immediate 1% rise in inflation – see the linearized form of the Phillips curves, (C.10) and (C.11).

in the retail sectors to give

$$
\begin{aligned}
Y_{F,t}(f) &= (1-c_i)F(A_{F,t}, N_{F,t}(f), K_{F,t}(f)) = \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\zeta_F} Y_{F,t} && (27)\\
Y_{I,t}(f) &= (1-c_i)F(A_{I,t}, N_{I,t}(f), K_{I,t}(f)) = \left(\frac{P_{I,t}(f)}{P_{I,t}}\right)^{-\zeta_I} Y_{I,t} && (28)
\end{aligned}
$$

using (23) and (24). Then solving for $N_{i,t}$, $i = F, I$ and defining aggregate employment-hours in each sector by $N_{i,t} = \sum_{j=1}^{\nu_i} N_{i,t}(j)$, $i = F, I$ we arrive to the aggregate production functions

$$
Y_{i,t} = \frac{(1-c_i)A_{i,t}N_{i,t}^{\alpha_i}K_{i,t}^{1-\alpha_i}}{\Delta_{i,t}}\ ;\ i = F, I \tag{29}
$$

where

$$
\Delta_{i,t} = \sum_{j=1}^{\nu_i}\left(\frac{P_{i,t}(f)}{P_{i,t}}\right)^{-\frac{\zeta_i}{\alpha_i}} \tag{30}
$$

is a measure of the price dispersion across firms in sector $i = F, I$. Then the aggregate equilibrium conditions in each retail sector are

$$
\begin{aligned}
Y_{F,t} &= C_{F,t} + I_t + G_t\\
(1 - S(X_t))I_t &= K_{t+1} - (1-\delta)K_t\\
K_t &= K_{F,t} + K_{I,t}\\
Y_{I,t} &= C_{I,t}
\end{aligned}
$$

with aggregate production functions (29).

Given government spending G_t, technology $A_{i,t}$, mark-up shocks $MS_{i,t}$, the nominal interest rate $R_{n,t}$, the real wage norm RW_t and choice of numeraire, the above system defines a general equilibrium in C_t, P_t, $P_{i,t}$, $P_{i,t}^W$, $C_{i,t}$, $h_{F,t}$, $h_{I,t}$, $W_{F,t}$, $W_{I,t}$, $n_{i,t}$, $Y_{i,t} = (1-c_i)Y_{i,t}^W$ and $\hat{P}_{i,t}$ for $i = I, F$.

3.1.6 Monetary Policy and Government Budget Constraint

Monetary policy is conducted in terms of the nominal interest rate $R_{n,t}$ set at the beginning of period t. The expected real interest rate over the interval $[t, t+1]$ is given by

$$
E_t[1 + R_{t+1}] = E_t\left[(1 + R_{n,t})\frac{P_t}{P_{t+1}}\right]
$$

In what follows we consider interest rate policy in the form of ad hoc Taylor-type rules, optimized Taylor rules, optimal commitment rules and finally discretionary policy.

Fiscal policy for the moment assumes a balanced budget constraint in which and employment tax on only formal firms, τ_t, finances government spending. This takes the form

$$
P_{F,t}G_t = P_t(n_{F,t}h_{F,t}\tau_{F,t} + n_{I,t}h_{I,t}\tau_{I,t})
$$

noting that government services are provided out of formal output. In general, a *tax rule*

$$\tau_{I,t} = k\tau_{F,t}\,;\; k \in [0,1]$$

allows for the possibility that *some* tax can be collected in the informal economy.

3.1.7 Price Dispersion

Finally for second-order terms that affect the welfare, we need to include price dispersion in the retail output. The production function for retail firm j in terms of its input of the homogeneous wholesale good is given by

$$Y_{i,t}(j) = (1-c_i)Y_{i,t}^W(j) = (1-c_i)(A_{i,t}N_{i,t}(j))^{\alpha_i}K_{i,t}(j)^{1-\alpha_i}\,;\; i = I, F \tag{31}$$

where $N_{i,t}(j)$ and $K_{i,t}(j)$ are labour and capital inputs into the wholesale sector required to ultimately produce the variety j. From (18) we have

$$1 + R_{i,t}^k = \frac{(1-\alpha_i)\frac{P_{i,t}^W}{P_t}\frac{Y_{i,t}^W(j)}{K_{i,t}(j)} + (1-\delta)Q_t}{Q_{t-1}}$$

where we note that the return $R_{i,t}^k$ and prices $P_{i,t}^W$, P_t and Q_t are all independent of j. From (3.1.7) we can write the capital-output ratio as

$$KY_{i,t} \equiv \frac{K_{i,t}(j)}{Y_{i,t}^W(j)} = \frac{P_{i,t}^W}{P_t}\left[\frac{(1+R_{i,t}^k)Q_{t-1} - (1-\delta)Q_t}{(1-\alpha_i)Q_{t-1}}\right] \tag{32}$$

which again is independent of j. Now we can write (31) as

$$Y_{i,t}(j) = (1-c_i)A_{i,t}N_{i,t}(j))KY_{i,t}^{\frac{1-\alpha_i}{\alpha_i}}\,;\; i = I, F$$

Using the Dixit-Stiglitz result for the demand for variety j:

$$Y_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\zeta_i}$$

and defining total labour supply for sector i as $N_{i,t} \equiv \sum_j N_{i,t}(j)$ we arrive at the aggregate production function

$$Y_{i,t} = \frac{(1-c_i)A_{i,t}N_{i,t}KY_{i,t}^{\frac{1-\alpha_i}{\alpha_i}}}{\Delta_{i,t}}\,;\; i = I, F \tag{33}$$

where

$$\Delta_{i,t} \equiv \sum_{j=1}^{\nu_i}\left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\zeta_i} \tag{34}$$

is a measure of the price dispersion across firms in sector $i = F, I$. This can be shown to be given by

$$\Delta_{i,t} = \xi_i \Pi_{i,t}^{\zeta_i} \Delta_{i,t-1} + (1-\xi_i) \left(\frac{J_{i,t}}{H_{i,t}} \right)^{-\zeta_i}$$

In fact writing aggregate wholesale output as

$$Y_{i,t}^W = (1-c_i)(A_{i,t} N_{i,t})^{\alpha_i} K_{i,t}^{1-\alpha_i} \; ; \; i = I, F \tag{35}$$

(33) simply becomes

$$Y_{i,t} = \frac{(1-c_i) Y_{i,t}^W}{\Delta_{i,t}} \; ; \; i = I, F$$

3.1.8 Functional Forms

We choose a Cobb-Douglas production function, AR(1) processes for labour-augmenting productivity (LAP), government spending and mark-up shocks and a utility function consistent with balanced growth:

$$\begin{aligned}
F(A_{i,t}, N_{i,t}) &= (A_{i,t} N_{i,t})^{\alpha_i} K_{i,t}^{1-\alpha_i} \\
\log A_{i,t} - \log \bar{A}_{i,t} &= \rho_{A_i} (\log A_{i,t-1} - \log \bar{A}_{i,t-1}) + \epsilon_{A_i,t} \\
\log G_t - \log \bar{G}_t &= \rho_G (\log G_{t-1} - \log \bar{G}_{t-1}) + \epsilon_{G,t} \\
U_t(C_t, L_{i,t}) &= \frac{[C_t^{1-\varrho} L_{i,t}^{\varrho}]^{1-\sigma} - 1}{1-\sigma} ; \quad \sigma > 1 \\
&= (1-\varrho) \log C_t + \varrho \log L_{i,t} \, ; \quad \sigma = 1 \qquad (36) \\
\log \left[\frac{\bar{A}_{i,t}}{\bar{A}_{i,t-1}} \right] &= \log \left[\frac{\bar{G}_t}{\bar{G}_{t-1}} \right] = 1 + g \\
S(X) &= \phi_X (X_t - (1+g))^2
\end{aligned}$$

where $\epsilon_{A_i,t}, \epsilon_{G_i,t}, \sim ID$ with zero mean. The choice of utility function in (36) is chosen to be consistent with a steady state balanced growth path (henceforth BGP) where LAP $\bar{A}_t$ and $\bar{G}_t$ are time-varying. As pointed out in Barro and Sala-i-Martin (2004), chapter 9, this requires a careful choice of the form of the utility as a function of consumption and labour effort. It is achieved by a utility function which is non-separable in consumption and leisure unless $\sigma = 1$. A utility function of the form (36) achieves this. The marginal utilities are then then given by

$$\begin{aligned}
\Lambda_{C,t} &= (1-\varrho) C_t^{(1-\varrho)(1-\sigma)-1} (n_{F,t} L_{F,t}^{\varrho(1-\sigma)} + (1-n_{F,t}) L_{I,t}^{\varrho(1-\sigma)}) \qquad (37) \\
U_{L_F,t} &= \varrho C_t^{(1-\varrho)(1-\sigma)} L_{F,t}^{\varrho(1-\sigma)-1} \\
U_{L_I,t} &= \varrho C_t^{(1-\varrho)(1-\sigma)} L_{I,t}^{\varrho(1-\sigma)-1}
\end{aligned}$$

3.2 The Model with Labour and Credit Market Frictions

We allow in general for the the real wage in the F-sector to be a combination of an exogenous real wage norm, RW_t and the market-clearing real wage in the I-sector:

$$\frac{W_{F,t}}{P_t} = RW_t > \frac{W_{I,t}}{P_t}$$

From $RW_t > \frac{W_{I,t}}{P_t}$, it follows from $U_{LL} < 0$ that the household will choose less leisure and more work effort in the F-sector; i.e., $h_{F,t} > h_{I,F}$. Household members would prefer to be employed in the F-sector at the higher wage rate; but all those who fail to do so find employment at the the market-clearing wage rate in the I-sector.

We can model the risk premium rigorously by introducing a financial accelerator in each sector. The main ingredient is an external finance premium $\Theta_{i,t} \geq 1$ such that the expected cost of borrowing in sector $i = I, F$ is given by

$$E_t[1 + R_{i,t+1}] \equiv E_t[\Theta_{i,t+1}(1 + R_{t+1})] \tag{38}$$

where

$$\Theta_{i,t} = s_i\left(\frac{N_{i,t}}{Q_{t-1}K_{i,t}}\right) \; ; \; s'(\cdot) < 0 \tag{39}$$

In (39), $N_{i,t}$ is net worth in sector $i$and $Q_{t-1}K_{i,t} - N_{i,t}$ is the external financing requirement. Thus $\frac{Q_{t-1}K_{i,t}-N_{i,t}}{N_{i,t}}$ is the leverage ratio and thus (38) and (39) state that the cost of capital is an increasing function of this ratio.

Assume that entrepreneurs in the sector i exit with a given probability $1 - \phi_i$. Then the net worth accumulates according to

$$N_{i,t+1} = \phi_i V_{i,t} + (1 - \phi_i) D_{i,t}$$

where $D_{i,t}$ are transfers from exiting to newly entering entrepreneurs continuing, and $V_{i,t}$, the net value carried over from the previous period, is given by

$$V_{i,t} \;\; = \;\; (1 + R^k_{i,t})Q_{t-1}K_{i,t} - \Theta_{i,t}(1 + R_t)(Q_{t-1}K_{i,t} - N_{i,t})$$

where $R^k_{i,t}$ is the ex post return on capital in the i-sector given by

$$1 + R^k_{i,t} \;\; = \;\; \frac{\frac{P^W_{i,t}}{P_t} F_{K_i} + (1 - \delta)Q_t}{Q_{t-1}}$$

Demand for capital in sector $i = I, F$ is then given by

$$E_t[1 + R_{i,t+1}] \;\; = \;\; \frac{E_t\left[\frac{P^W_{i,t+1}}{P_{t+1}} F_{K_i,t+1} + (1 - \delta)Q_{t+1}\right]}{Q_t} \equiv E_t[1 + R^k_{i,t+1}]$$

Finally exiting entrepreneurs consume the residual equity so that their consumption

$$C^e_{i,t} = \frac{1-\phi_i}{\phi_i} N_{i,t}$$

must be added to total consumption.

We choose the following functional form for $s(\cdot)$:

$$s_i\left(\frac{N_{i,t}}{Q\bar{K}_{i,t}}\right) = RPS_t k_i \left(\frac{N_{i,t}}{Q\bar{K}_{i,t}}\right)^{-\chi_i}$$

where RPS_t is an exogenous risk premium shock that is common to both sectors.[9]

4 Steady State Analysis and Model Calibration

The zero-inflation balanced-growth deterministic steady state of the model economy is given by

$$\frac{\bar{\Lambda}_{C,t+1}}{\bar{\Lambda}_{C,t}} \equiv 1 + g_{\Lambda_C} = \left[\frac{\bar{C}_{t+1}}{\bar{C}_t}\right]^{(1-\varrho)(1-\sigma)-1} = (1+g)^{((1-\varrho)(1-\sigma)-1)}$$

using (37). Thus from (5)

$$1 + R_n = 1 + R = \frac{(1+g)^{1+(\sigma-1)(1-\varrho)}}{\beta}$$

$$\begin{aligned}
n_I + n_F &= 1 \\
P &= \left[\mathrm{w}(P_F)^{1-\mu} + (1-\mathrm{w})(P_I)^{1-\mu}\right]^{\frac{1}{1-\mu}} \\
\bar{Y}_{i,t} &= (1-c_i)(n_i h_i \bar{A}_{i,t})^{\alpha_i} \bar{K}^{1-\alpha_i}_{I,t}\,;\ i = F, I \\
\frac{\varrho \bar{C}_t}{(1-\varrho) L_F \left(n_F + (1-n_F)\left(\frac{L_F}{L_I}\right)^{\varrho(\sigma-1)}\right)} &= \bar{W}_{F,t} \\
\frac{\varrho \bar{C}_t}{(1-\varrho) L_I \left(n_F \left(\frac{L_I}{L_F}\right)^{\varrho(\sigma-1)} + 1 - n_F\right)} &= \bar{W}_{I,t} \\
\frac{\alpha_i P^W_I \bar{Y}^W_{i,t}}{P n_i h_i} &= \bar{W}_{i,t} + \bar{\tau}_{i,t} \equiv \bar{W}_{i,t}(1+\tau_{i,t})\,;\ i = F, I \\
\bar{W}_{F,t} &= (R\bar{W}_t)^{1-\omega} (\bar{W}_{I,t})^{\omega}
\end{aligned}$$

[9] In a subsequent comparison of the models with and without financial frictions we will retain this shock in the latter model as a 'real interest rate shock'.

$$
\begin{aligned}
\frac{P\bar{K}_{i,t}}{P_i^W \bar{Y}_{i,t}^W} &= \frac{1-\alpha_i}{R_i+\delta}\,;\ i=I,F \\
\bar{I}_t &= (\delta+g)(\bar{K}_{I,t}+\bar{K}_{F,t}) \\
\bar{N}_{i,t} &= \frac{(1-\phi_i)\bar{D}_{F,t}}{(1-\phi_i(1+R_i))}\,;\ i=I,F \\
1+R_i &= (1+R)\ s_i\left(\frac{\bar{N}_{i,t}}{\bar{K}_{i,t}}\right)\,;\ i=I,F \\
Q &= 1 \\
\bar{C}_{i,t}^e &= \frac{1-\phi_i}{\phi_i}\bar{N}_{i,t}
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\bar{Y}_{I,t} &= \bar{C}_{I,t} = (1-\mathrm{w})\left(\frac{P_I}{P}\right)^{-\mu}(\bar{C}_t+\bar{C}_{I,t}^e+\bar{C}_{F,t}^e) \\
\bar{Y}_{F,t} &= \bar{C}_{F,t}+\bar{G}_t = \mathrm{w}\left(\frac{P_F}{P}\right)^{-\mu}(\bar{C}_t+\bar{C}_{I,t}^e+\bar{C}_{F,t}^e)+\bar{I}_t+\bar{G}_t \\
\frac{P_F}{P}\bar{G}_t &= n_F h_F \bar{\tau}_{F,t}+n_I h_I \bar{\tau}_{I,t} \\
\bar{\tau}_{i,t} &= \tau_i \bar{W}_{i,t}\,;\ i=F,I \\
\bar{\tau}_{I,t} &= k\bar{\tau}_{F,t} \\
P_i &= \frac{1}{1-\frac{1}{\zeta_i}}P_i^W
\end{aligned}
$$

where consumption, technical LAP, the real wage and tax rates, and government spending (all indicated by $\bar{X}_t$) are growing at a common growth rate.

We impose a free entry condition on retail firms in this steady state which drives monopolistic profits to zero. This implies that costs of converting wholesale to retail goods are given by

$$c_i = 1/\zeta_i$$

which implies that:

$$P_i\bar{Y}_{i,t} = P_i^W \bar{Y}_{i,t}^W\,;\ i=F,I$$

Given exogenous trends for $\bar{A}_{i,t}$ and $\bar{G}_t$, the tax rates and RW_t, the above system of equations give 21 relationships in 22 variables R, P, P_F, P_I, P_F^W, P_I^W, $\bar{C}_t$, $\bar{C}_{F,t}$, $\bar{C}_{I,t}$, $\bar{Y}_{F,t}$, $\bar{Y}_{I,t}$, $\bar{W}_{I,t}$, $\bar{W}_{F,t}$, n_I, n_F, h_I, h_F, $\bar{I}$, $\bar{K}_F\,\bar{K}_I$, $\bar{\tau}_{F,t}$, $\bar{\tau}_{I,t}$. One of the prices (it is convenient to choose P) can be chosen as the numeraire, so the system is determinate.

4.1 Model Calibration

Turning to the calibration, the idea is to assume an observed baseline steady state equilibrium in the presence of some observed policy. We then use this observed equilibrium to solve for model parameters consistent with this observation For this baseline and for the purpose of calibration only, it is convenient to choose units of retail output such that their prices are unitary; i.e., $P_F = P_I = 1$. Then assuming $\zeta_F = \zeta_I$, from (4)

$$P = \left[\frac{\mathrm{w}}{\left(1 - \frac{1}{\zeta_F}\right)^{1-\mu}} + \frac{(1-\mathrm{w})}{\left(1 - \frac{1}{\zeta_I}\right)^{1-\mu}} \right]^{\frac{1}{1-\mu}} = 1$$

We can choose units of labour supply h_I so that $\bar{A}_I = 1$, but the choice of A_F must be consistent with the choice of unitary prices.

Our baseline steady state can be described in terms of a vector $\underline{X} = f(\underline{\theta})$ of outcomes where $\underline{\theta}$ is a vector of parameters. The calibration strategy is to choose a subset $\underline{X}_1$ of n observed outcomes to calibrate a subset $\underline{\theta}_1$ of n parameters. Partition $\underline{X} = [\underline{X}_1, \underline{X}_2]$ and $\underline{\theta} = [\underline{\theta}_1, \underline{\theta}_2]$. Then $\underline{\theta}_1$ is then found by solving

$$[\underline{X}_1, \underline{X}_2] = f([\underline{\theta}_1, \underline{\theta}_2])$$

for $\underline{X}_2$ and $\underline{\theta}_1$, given $\underline{X}_1$ and $\underline{\theta}_2$. If such a solution exists for economically meaningful parameter values for $\underline{\theta}_1$ then a successful calibration has been achieved.

We now calibrate the parameters $\underline{\theta}_1 = [\varrho, \mathrm{w}, \beta, \alpha_I]$ given observations $\underline{X}_1 = [n_F, \frac{\bar{Y}_{F,t}}{\bar{Y}_{I,t}}, h_F, R]$. The given the government share of formal output g_{yF}, $\bar{G}_t$ is given by $\frac{\bar{G}_t}{\bar{Y}_{F,t}} \equiv g_{yF}$. $\bar{A}_F$ is then determined by the normalization of unitary prices. Imposed parameters found from micro-econometric studies are σ, rw and δ and the long-run growth rate g chosen to correspond to 5% per year. The results of this calibration are summarized in Table 1.

For the financial accelerator, we choose a functional form for sector $i = I, F$:

$$\Theta_i = s_i \left(\frac{N_{i,t}}{Q\bar{K}_{i,t}} \right) = k_i \left(\frac{N_{i,t}}{Q\bar{K}_{i,t}} \right)^{-\chi_i} \tag{41}$$

Suppose we can obtain χ_i from econometric studies and we have data on the risk premium $\Theta_i = \frac{1+R_i}{1+R}$ and leverage (= borrowing/net worth) in both sectors

$$\ell_i = \frac{QK_i - N_i}{N_i} = \frac{QK_i}{N_i} - 1 = \frac{1}{n_{k,i}} - 1$$

defining $n_{k,} \equiv \frac{N_i}{QK_i}$. Then we can set the scaling parameter k_i from (41) as

$$k_i = \Theta n_{k,i}^{\chi_i}$$

Then in the baseline steady state used to calibrate parameters, we put $\bar{N}_{i,t} = n_{k,i}\bar{K}_{i,t}$ and calibrate $\bar{D}_i$ from (40).

Data on emerging economies can be obtained from IMF, World Bank and ILO statistics. As discussed in Neumeyer and Perri (2004) real interest rates in emerging economies are very volatile and difficult to calculate. Though nominal interest rate statistics are usually reported by local Central Banks, due to the high variability of inflation in emerging economies, the calculation of the real interest rate in EMEs countries is often cumbersome and not reliable. For this reason the authors calculate the real interest rate from a combination of government bonds (Argentina) and dollar denominated bonds index (EMBI) constructed by J.P. Morgan. In this way real interest rates can be computed without relying on expected inflation rates. For example, the average real interest rate per year for Argentina is 14.5%. We choose a value 8% for our calibration. Also GDP growth rates differ considerably among EMEs countries. Neumeyer and Perri (2004) reports an annual growth rate for Argentina of 2.5% in a period range from 1984 to 2002. We can use this average growth rate and consider Argentina our representative country or calculate an average among EMEs in Latin America, Asia and Transitional Economies. The World Development Indicators published by the World Bank shows an annual growth rate for 2007 of 5.8% for Latin American countries, 9.1% for India and 6% for the Czech Republic (average selected EMEs countries = 7%). We choose a value of GDP annual growth rate of 6% keeping in mind there are large differences among EMEs countries and period considered. [10] growth rate We refer to LABSTAT (ILO) for the calculation of hours of work in emerging economies and choose h=45/100 and to ILO (2002) for data on formal and informal output $rel^{obs} = 7/3$ as reported in table 2.8 of the ILO's report. Data on government shares can be obtained from different sources such as IMF and World Bank. We choose World Bank and calculate an average for selected EMEs countries to obtain a value equal to 15%. [11] For values of wage mark-up in the formal sector, we refer to Perry *et al.* (2007) where Latin American data are reported. Table 3.1 shows that, on average, informal salaried workers earn between 40 to 66 percent less than formal salaried workers. Looking at this figures, we choose a mark-up of 50%. Finally, data on the formal sector employment as reported in various ILO's documents range from 60 percent to 35% in selected EMEs countries with a particular low level of 15 percent in India. We choose a value of 50% which is also consistent with Sparts (2003) for Bolivia (table 4).

[10] Note that this is less than the steady-state real interest rate ensuring a binding intertemporal solvency constraint for households and government.

[11] In general, government spending in emerging economies is lower than the one in developed economies.

Imposed Parameters	Value
δ	0.025
σ	2.0
$\zeta_F = \zeta_I$	7.0
μ	1.5
g	0.015
α_F	0.5
rw	0.5
g_{yF}	0.15
ξ_F	0.75
ξ_I	0.50
ℓ_F	1.0
ℓ_I	0.33
$PREM_I \equiv R_I - R$	0.057
$PREM_F \equiv R_F - R$	0.012
$\chi_F = \chi_I$	0.1
ϕ_F	0.96
ϕ_I	0.93
Observed Equilibrium	Value
$P_F = P_I = P$	1
n_F	0.5
h_F	0.45
$rel \equiv \frac{P_F Y_F}{P_I Y_I}$	7/3
R	0.02
Calibrated Parameters	Value
α_I	0.6821
β	0.9881
w	0.6304
ϱ	0.6087
Rest of Equilibrium	Value
$\frac{K_F}{Y_F}$	10.1742
$\frac{K_I}{Y_I}$	3.2528
$\frac{I}{Y_F}$	0.2815
$\frac{C_F}{Y_F}$	0.5685
$\frac{C_F}{C_I}$	1.7056
h_I	0.2923

Table 1. Calibration

This completes the calibration of the parameters describing the deterministic parameters. There are currently two exogenous shocks in the model to labour productivity in both sectors and government spending. In the linearized model of Appendix A these are denoted respectively by $a_{i,t}$ and $g_{i,t}$, $i = I, F$. We also add mark-up shocks to the linearized Phillips Curves $u_{i,t}$, $i = I, F$. Again following the literature we assume AR(1) processes with calibrated persistence parameters 0.7 for the technology and demand shocks. Mark-up shocks are assumed to be transient. The standard deviations of the innovation processes are taken to be unity, but later we examine more volatile economies with a standard deviation $k > 1$. This completes the calibration; observations, imposed and calibrated parameters are summarized in Table 1.

Figure 2 shows this process of informalization for different degrees of wage rigidity and illustrates how an increase in this friction coupled with a lack of tax-smoothing drives down the size of the formal sector. For example, with $k = 0$ and no friction the size of the formal sector is close to $n_F = 0.82$. When $rw = 0.75$, this halves, falling to under $n_F = 0.4$.

Figure 3 shows the welfare effects on a representative household as the tax burden is smoothed over the two sectors. As k approaches unity the utility becomes very flat and close to the optimum. We can work out the equivalent permanent increase in consumption implied by this optimum by first computing the increase from a 1% consumption change at any point on the balanced growth trend as $n_F U(1.01 \times \bar{C}_t, L_F) + (1 - n_F) U(\bar{C}_t, L_I)$ at some time $t = 0$ say. In our best steady state equilibrium for $rw = 0.5$ at $k = 1$, this works out as 0.0059, so any increase in welfare $D\Lambda$ implies a consumption equivalent $c_e = \frac{D\Lambda}{0.0059}\%$. For the transition between $k = 1$ to $k = 0$ with $rw = 0.5$ in Figure 2 this implies a utility loss $c_e = 4.38\%$.

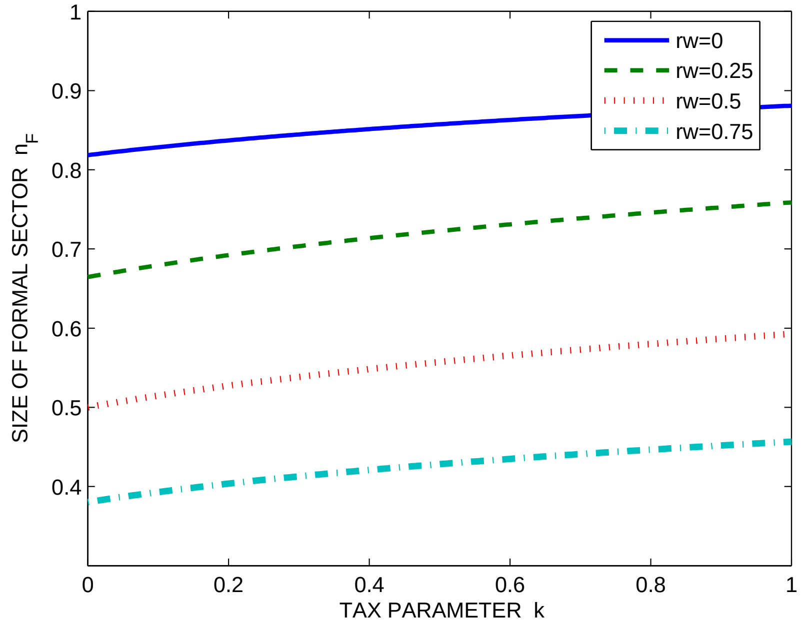

Figure 2: **The Size of Formal Sector and Tax Burden: k = Ratio of Informal-Formal Tax Rates. rw =wage mark-up in the formal sector.**

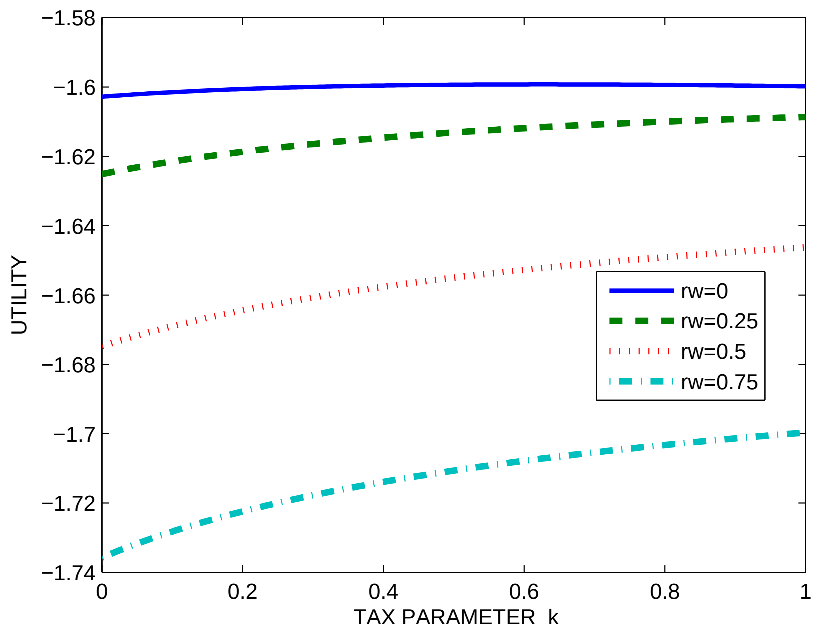

Figure 3: **Welfare and Tax Burden: k = Ratio of Informal-Formal Tax Rates. rw =wage mark-up in the formal sector.**

5 Optimal Stabilization Policy and Potential Gains from Commitment

We adopt a linear-quadratic framework for the optimization problem facing the monetary authority. This is particularly convenient as we can then summarize outcomes in terms of unconditional (asymptotic) variances of macroeconomic variables and the local stability and determinacy of particular rules. The framework also proves useful for addressing the issue of the zero lower bound on the nominal interest rate.

We pose the central question: *how does informality affect monetary policy*? To answer this question we compare outcomes under welfare-optimal monetary policy, discretion and welfare-optimized Taylor rules in three models: first as a benchmark we examine our two-sector NK model with no frictions in labour and credit markets, **model I**. We then proceed to **model II** with frictions in only the formal labour market and finally **model III** with frictions on both credit markets and the formal labour market.

5.1 LQ Approximation of the Optimization Problem

In our models there are three sets of distortions that result in the steady state output being below the social optimum: namely, from monopolistic competition, from distortionary taxes and from the labour and credit frictions. We cannot assume that these distortions are small in the steady state and use the 'small distortions' (Woodford (2003)), quadratic approximation to the household's single period utility which is accurate in the vicinity of our zero-inflation steady state. Our computations use the large distortions approximation to this welfare function as described in Levine *et al.* (2008a) and summarized in Appendix C. However it is instructive to see the form of the 'small distortions' approximation to the loss function in a simpler case of the model without capital.

The small distortions approximation is found by approximating the utility function $U_t = U(C_t, L_t)$ in consumption, C_t and leisure $L_t = 1 - h_t$ subject to the resource constraint. We start with the Taylor Series expansion about the BGP steady state[12]

$$U_t = U + U_C C c_t + \frac{1}{2} U_{CC} C^2 c_t^2 + U_L L l_t + \frac{1}{2} U_{LL} L^2 l_t^2 + \text{higher order terms} \tag{42}$$

Next we write $c_t = \mathrm{w} c_{F,t} + (1-\mathrm{w}) c_{I,t}$, $l_t = -\frac{h}{1-h}\hat{h}_t$ and use the linearized resource constraints

$$\begin{aligned} y_{F,t} &= a_{F,t} + \alpha_F(\hat{n}_{F,t} + \hat{h}_t) - d_{F,t} = (1 - g_{Fy})c_{F,t} + g_{Fy} g_t \\ y_{I,t} &= a_{I,t} + \alpha_F(\hat{n}_{I,t} + \hat{h}_t) - d_{I,t} = c_{I,t} \end{aligned}$$

[12]The BGP is time-varying but here we drop the bar and time-script in $\bar{U}_t, \bar{C}_t$ etc.

where

$$d_{i,t} = \log\left[\frac{\Delta_{i,t}}{\Delta_i}\right] ; i = I, F \tag{43}$$

and $\Delta_{i,t}$ is the price dispersion effect given by (34). By standard results (see, for example, Gali (2008), p88) $d_{i,t}$ is a *second order* term given by

$$d_{i,t} = \frac{\zeta_i(\alpha_i + (1-\alpha_i)\zeta_i)}{2\alpha_i}\mathrm{var}(p_{i,t}(j))\, ; i = I, F \tag{44}$$

and

$$\sum_{t=0}^{\infty}\beta^t \mathrm{var}(p_{i,t}(j)) = \frac{\xi_i}{(1-\beta\xi_i)(1-\xi_i)}\sum_{t=0}^{\infty}\beta^t\pi_{i,t}^2\, ; i = I, F \tag{45}$$

Then using the linearized resource constraints and the properties of efficiency in the steady state: $\frac{U_L}{U_C} = F_{N_F} = F_{N_I}$ the first order terms in c_t and l_t disappear in (42) and we are left the quadratic approximation to the utility function

$$U_t = U + U_C C\left[-\frac{\mathrm{w}}{(1-g_{Fy})}d_{F,t} - (1-\mathrm{w})d_{I,t}\right] + \frac{1}{2}U_{CC}C^2c_t^2 + U_L L l_t + \frac{1}{2}U_{LL}L^2 l_t^2 + \text{higher order terms} \tag{46}$$

Finally using the results (43)–(46) we can write the quadratic form of the intertemporal expected welfare loss at time $t = 0$ as

$$\Omega_0 = \frac{1}{2}E_t\left[\sum_{t=0}^{\infty}\beta^t[w_c c_t^2 + w_{hI}\hat{h}_I^2 + w_{hF}\hat{h}_F^2 + w_{\pi F}\pi_{F,t}^2 + w_{\pi I}\pi_{I,t}^2]\right] \tag{47}$$

where for our choice of utility function (36)

$$\begin{aligned}
w_c &= -\frac{U_{CC}C}{U_C} = 1 + (\sigma-1)(1-\varrho) \\
w_h &= -\frac{U_{LL}h^2}{U_C C} = \frac{(1+\varrho(\sigma-1))h^2}{(1-\varrho)(1-h)^2}\, ;\; h = h_I, h_F \\
w_{\pi F} &= \mathrm{w}\frac{\zeta_F(\alpha_F + (1-\alpha_F)\zeta_F)}{c_{Fy}\alpha_F\lambda_F} \\
w_{\pi I} &= \frac{(1-\mathrm{w})\zeta_I(\alpha_I + (1-\alpha_I)\zeta_I)}{\alpha_I\lambda_I} \\
\lambda_i &= \frac{\xi_i}{(1-\beta\xi_i)(1-\xi_i)}\, ; i = F, I
\end{aligned}$$

To work out the welfare in terms of a consumption equivalent percentage increase, expanding $U(C, L)$ as a Taylor series, a 1% permanent increase in consumption of 1 per cent yields a first-order welfare increase $U_C C \times 0.01$. Since standard deviations are expressed in terms of percentages, the welfare loss terms which are proportional to the covariance matrix (and pre-multiplied by 1/2) are of order 10^{-4}. The losses reported in the paper in the subsections that follow are scaled by a factor $1-\beta$. Letting $\Delta\Omega$ be these losses relative to the optimal policy, then $c_e = \Delta\Omega \times 0.01\%$.

5.2 Imposition of the ZLB Constraint

We can modify welfare criterion so as to approximately impose an interest rate zero lower bound (ZLB) so that this event hardly ever occurs. Our quadratic approximation to the single-period loss function can be written as $L_t = \mathrm{y}_t' Q \mathrm{y}_t$ where $\mathrm{y}_t' = [\mathrm{z}_t', \mathrm{x}_t']'$ and Q is a symmetric matrix. As in Woodford (2003), chapter 6, the ZLB constraint is implemented by modifying the single period welfare loss to $L_t + w_r r_{n,t}^2$. Then following Levine *et al.* (2008b), the policymaker's optimization problem is to choose w_r and the unconditional distribution for $r_{n,t}$ (characterized by the steady state variance) shifted to the right about a new non-zero steady state inflation rate and a higher nominal interest rate, such that the probability, p, of the interest rate hitting the lower bound is very low.[13] This is implemented by calibrating the weight w_r *for each of our policy rules* so that $z_0(p)\sigma_r < R_n$ where $z_0(p)$ is the critical value of a standard normally distributed variable Z such that prob $(Z \leq z_0) = p$, $R_n = \frac{1}{\beta(1+g_{u_c})} - 1 + \pi^* \equiv R_n(\pi^*)$ is the steady state nominal interest rate, $\sigma_r^2 = \mathrm{var}(r_n)$ is the unconditional variance and π^* is the new steady state inflation rate. Given σ_r the steady state positive inflation rate that will ensure $r_{n,t} \geq 0$ with probability $1 - p$ is given by[14]

$$\pi^* = \max[z_0(p)\sigma_r - R_n(0) \times 100, 0] \tag{48}$$

In our linear-quadratic framework we can write the intertemporal expected welfare loss at time $t = 0$ as the sum of stochastic and deterministic components, $\Omega_0 = \tilde{\Omega}_0 + \bar{\Omega}_0$. Note that $\bar{\Omega}_0$ incorporates in principle the new steady state values of all the variables; however the NK Phillips curve being almost vertical, the main extra term comes from the $\pi_{F,t}^2$ and $\pi_{I,t}^2$ terms in the loss function seen clearly in (47) for the small distortions case. By increasing w_r we can lower σ_r thereby decreasing π^* and reducing the deterministic component, but at the expense of increasing the stochastic component of the welfare loss. By exploiting this trade-off, we then arrive at the optimal policy that, in the vicinity of the steady state, imposes the ZLB constraint, $r_t \geq 0$ with probability $1 - p$.

5.3 Gains from Commitment Compared Across Models

We first assess the potential (maximum) gains from commitment by comparing the optimal commitment policy with discretionary policy, both subject to the constraint that the ZLB

[13]The idea that the ZLB should be avoided by choosing a long-run inflation rate rate so as increase the corresponding long-run interest rate and make room for an active interest rate rule at all times has been put forward recently by Blanchard *et al.* (2010).

[14]If the inefficiency of the steady-state output is negligible, then $\pi^* \geq 0$ is a credible new steady state inflation rate. Note that in our LQ framework, the zero interest rate bound is very occasionally hit in which case the interest rate is allowed to become negative.

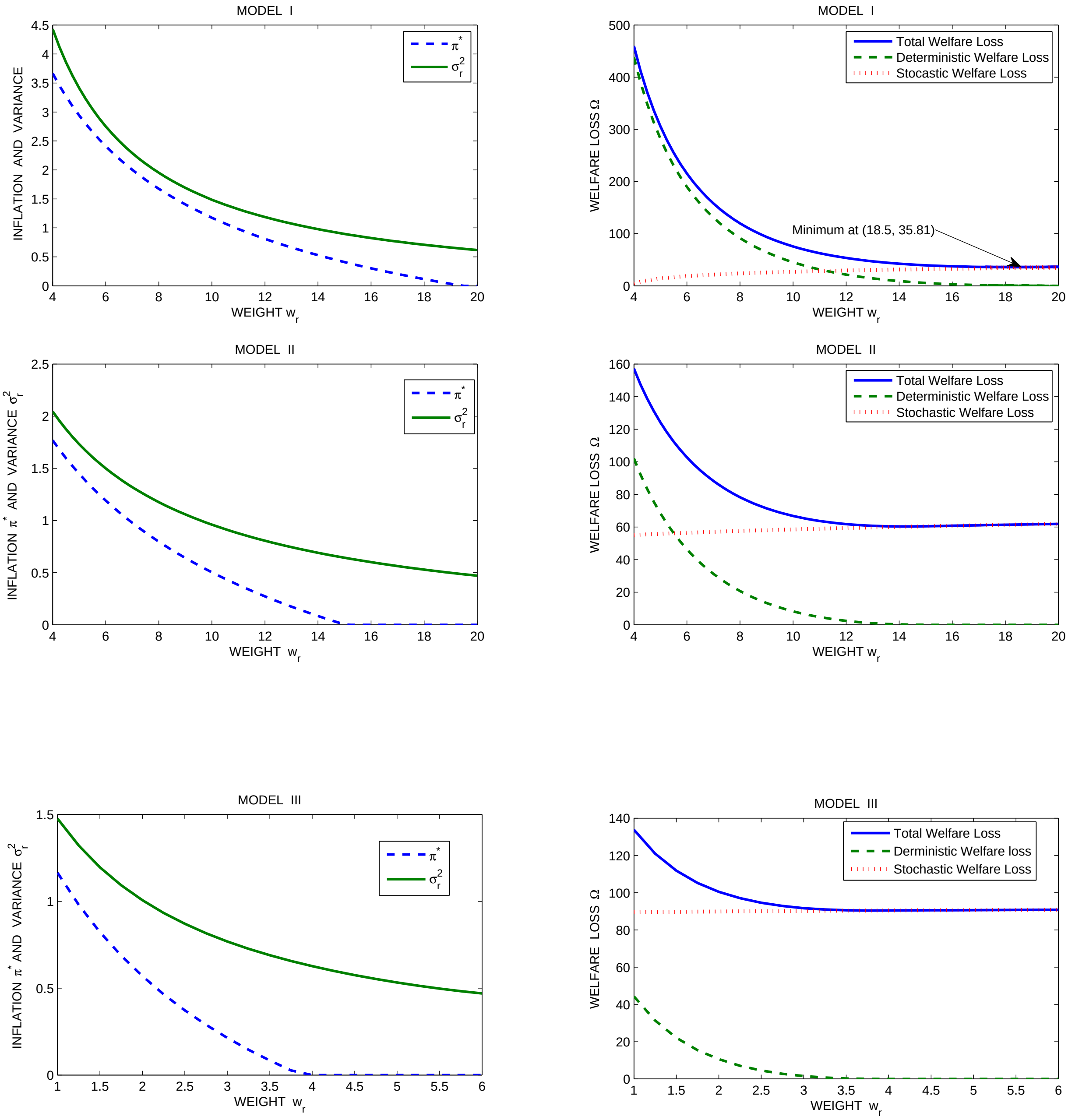

Figure 4: **Commitment: Imposing the ZLB Constraint**

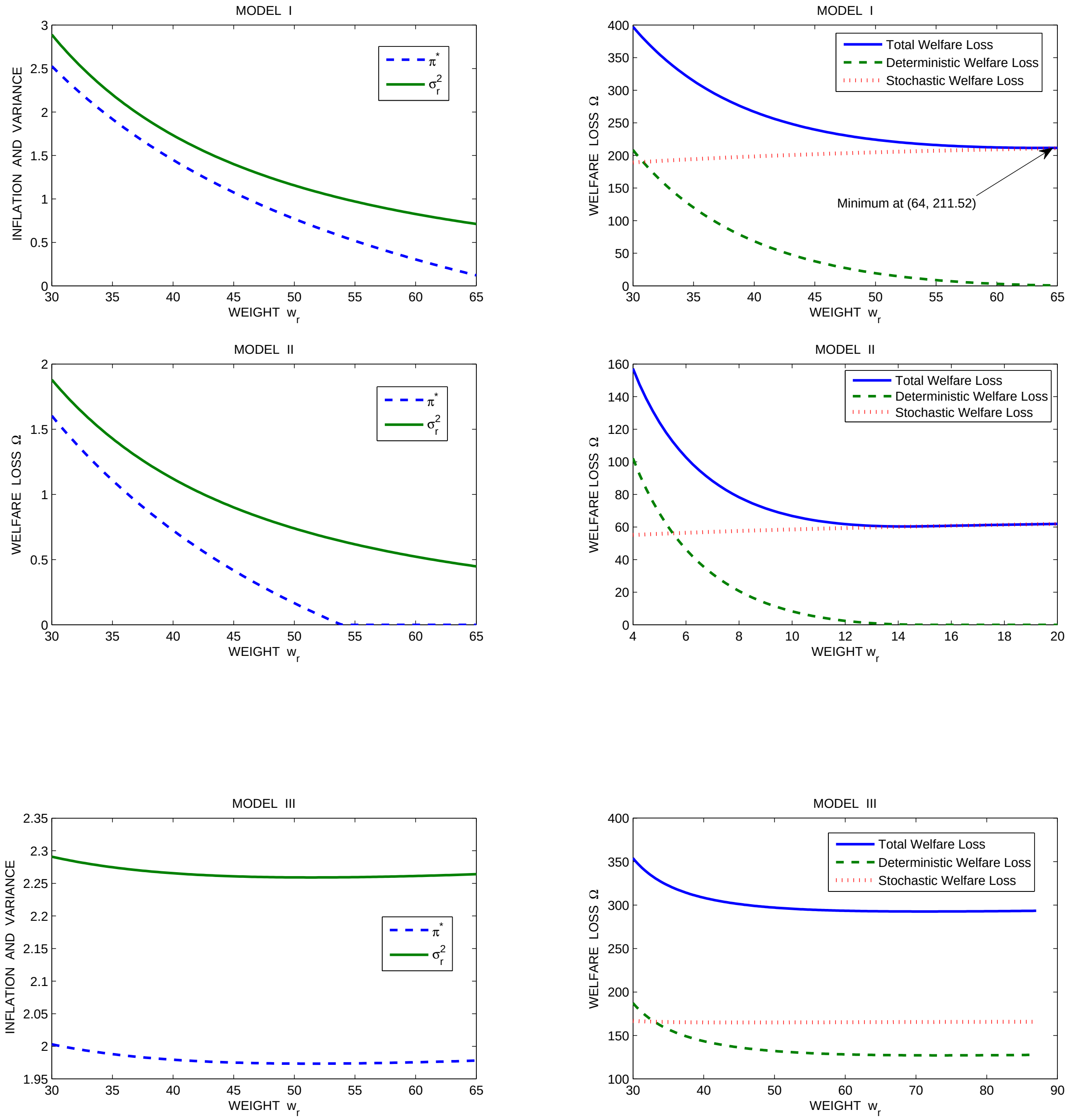

Figure 5: **Discretion: Imposing the ZLB Constraint**

is reached with a very small probability p. In a quarterly model we choose $p = 0.0025$ or an expected frequency of hitting the ZLB every 100 years. All shocks are AR(1) processes and we calibrate their standard deviations in deviation form about the steady state as 1% and the persistence parameters as 0.5.

Figures 4 and 5 show how the ZLB is imposed for commitment and discretion. In the figures on the lhs we see that as the penalty on the nominal interest rate volatility w_r in the single-period loss function is increased, then unconditional variance σ_r^2 falls and with it the steady-state inflation rate π^* given by (48) also falls. The graphs on the rhs show the rising stochastic loss function that subtracts the interest rate penalty how, the falling deterministic loss function arising from the positive steady state inflation rate, and the sum of these two, Ω_0, that falls and eventually reaches a minimum. This minimum, together with σ_r^2 and π^* is shown in table 3 for models I, II and III for both commitment and discretion. The consumption equivalent percentage loss c_e relative to the best outcome, the model I commitment case, is also computed.

Model	Policy	Frictions	Ω_0	σ_r^2	π^*	$c_e(\%)$
I	Commit	Product	35.8	0.684	0.074	0
I	Discretion	Product	211.5	0.734	0.155	1.76
II	Commit	Product, Labour	60.3	0.678	0.064	0.25
II	Discretion	Product, Labour	185.0	0.680	0.068	1.49
III	Commit	Product, Labour, Credit	90.4	0.657	0.027	0.55
III	Discretion	Product, Labour, Credit	292.6	2.26	1.97	2.56

Table 2. Optimal Rules with and without Commitment

A number of notable results emerge from table 2. First, as noted in Levine *et al.* (2008b) the gains from commitment with ZLB considerations are significant and vary between $c_e = 1.76\%$ in model I with no labour or financial market frictions to $c_e = 2.26\%$ in model III with such frictions. Interestingly in model II where the real wage in the F-sector is fixed at ite real norm the gains from commitment actually fails. The intuition here is straightforward: with the real wage in the F-sector fixed, the incentive to tackle the product market friction by engaging in a surprise inflation that lowers the real wage and increases output in the vicinity of a steady state that is below the efficient output level is less than in model I. In model III with credit market frictions the inability of the policymaker lacking commitment and the ability to influence private sector behaviour with promises of future interest rate changes becomes critical. The time inconsistency problem worsens and with it the gain from commitment. The steady state inflation rate needed to give room for interest rate changes rises to almost 2% per quarter, a level of inflation that is typical of emerging economies. It should be stressed that these commitment gains are for shocks with standard deviations all calibrated at 1%. If shocks are say 2%, then the consumption equivalent figures will increase by *at least* fourfold, when the ZLB constraint is imposed. Similarly if the persistence of shocks increases from it present calibration of 0.5 to 0.75, then welfare gains increase by at least a factor $\frac{1-0.5^2}{1-0.75^2} = 1.714$, so with a not implausible calibration, these commitment gains can become quite considerable, especially for model III.

Model	$a_{F,t}$ (%)	$a_{I,t}$ (%)	g_t (%)	$ms_{F,t}$ (%)	$ms_{I,t}$ (%)	rps_t (%)	Total (%)
I	2.15	0.56	0.19	88.47	1.19	7.42	100
II	1.69	0.42	0.17	92.06	0.87	4.77	100
III	1.12	0.31	0.10	86.31	0.01	11.27	100

Table 3. Welfare Decomposition under Optimal Policy

Finally Table 3 shocks how six exogenous shocks contribute to the welfare loss under optimal policy for the three models. By far the most important shock in this respect is the mark-up shock in the F-sector, $ms_{F,t}$, then comes the risk premium shock rps_t followed by the technology shock in the F-sector $a_{F,t}$. This decomposition contrasts with the RBC model which is at the core of our NK model and arrived at by stripping the latter of its nominal features with price stickiness. It is the latter and its effect on welfare that leads to dominant effect of mark-up shocks that in effect are shocks to inflation.

6 Optimized Taylor Commitment Rules

We consider *symmetrical* and *asymmetrical* interest rate Taylor commitment rules that respond to deviations of inflation and the output gap[15] in both formal and informal sectors and we allow for a degree of interest rate smoothing. Symmetrical rules are hypothetical in the sense that they require the full visibility of inflation and output in the I-sector. Asymmetrical rules require only data for F-sector output and aggregate inflation and are implementable. The comparison between these two forms of rules enable us to assess the welfare costs of the lack of visibility for the I-sector and these are part of the costs of

[15] In fact our measure of the output gap is simply the log-deviation about the steady state. We choose this form of rules to avoid problems the monetary would have observing the true output gap. In fact some experimentation suggests using the latter would make little difference to our results.

informality.[16] We write the rules in log-linear form as:

$$r_{n,t} = \rho r_{n,t-1} + \theta_{\pi_F}\pi_{F,t} + \theta_{\pi_I}\pi_{I,t} + \theta_{Iy}y_{I,t}\,;\ \theta_{\pi_F},\,\theta_{\pi_I},\,\theta_{Fy},\,\theta_{Iy} > 0 \tag{49}$$

$$r_{n,t} = \rho r_{n,t-1} + \theta_{\pi}\pi_{t} + \theta_{Fy}y_{F,t}\,;\ \theta_{\pi} > 0 \tag{50}$$

and we compute optimal parameter values that optimize Ω_0.

We now wish to address three questions. First can the optimized Taylor rule mimic the fully optimal commitment rule? Second, what are the costs of being constrained to an implementable asymmetric rule? Third, what form do the optimized rules take as we proceed from model I with no labour and credit market frictions to model III with these features? The results for the optimized Taylor rule are displayed in Table 4. The consumption equivalent changes in utility are again measured relative to the best outcome which is the optimal policy in model I. Thus $c_e = 0$ in this case.

Model	Rule	$[\rho,\ \theta_{\pi F},\ \theta_{\pi I},\ \theta_{yF},\ \theta_{yI}]$	Ω_0	σ_r^2	π^*	$c_e(\%)$
I	Sym TR	[1.00, 0.538, 0.000, 0.270, 0.161]	59.4	0.12	0	0.24
I	Asym TR	[0.953, 0.024, 0, 0.024, 0]	115.7	0.01	0	0.80
I	Optimal	Complex	35.8	0.684	0.074	0
II	Sym TR	[1.00, 0.428, 0.013, 0.400, 0.642]	86.4	0.14	0	0.51
II	Asym TR	[0.663, 0.337, 0, 0.000, 0]	127.8	0.42	0	0.92
II	Optimal	Complex	185.0	0.680	0.068	0.25
III	Sym TR	[0.732, 0.321, 0.027, 0.069, 0.042]	101.4	0.69	0.083	0.66
III	Asym TR	[0.871, 0.327, 0, 0.022, 0]	132.5	0.70	0.102	0.97
III	Optimal	Complex	90.4	0.657	0.027	0.55

Table 4. Optimized Taylor Rules Compared with Optimal Policy

The first result that emerges from table 4 is that the ZLB constraint does not bind for our simple rules for models I and II without financial frictions. Simplicity restricts the monetary authority from responding to all aspects of the economy and so keeps interest rate volatility low, at least for models I and II. In response to the questions we have posed, the costs of simplicity are measures as the differences between the optimal and simple rules for each model. For the hypothetical symmetrical rule these are in the range $0.11--0.24\%$ of consumption equivalent but rise to $0.42--0.80\%$ for implementable asymmetrical rules. Interestingly the costs do not rise as frictions are added, and in fact fall. But again these numbers depend on the calibration of the AR(1) shocks where modest standard deviations

[16]See Batini/etal2010 for a full assessment of the costs and benefits of informality.

of 1% and persistence parameters of 0.5 were chosen. A plausible higher choice of these values would see these costs rising by at least a factor $4 \times 1.71 = 6.8$ so costs of simplicity and lack of visibility of the I-sector can be very high.

Turning to the form of the optimized implementable Taylor rules and how they change with added frictions. Comparing the asymmetrical rules across the models it is of interest to examine the long-term responses to aggregate inflation, $\frac{\theta_{pi}}{1-\rho}$ and to output in the F-sector, $\frac{\theta_y}{1-\rho}$. For model I these are 0.51 for both so that the *Taylor principle is violated* - ceteris paribus real interest rate fall with a rise in inflation - but the economy is stabilized by allowing the nominal interest rate to respond equally strongly to a rise in output. When we introduce labour market frictions in model II the long-run responses to inflation and output are 1.003 and 0.000 so the Taylor principle now just holds and there is no significant response to output. Then adding financial frictions in model III the corresponding responses are 2.53 and 0.17 so now the Taylor principle is easily satisfied and accompanied by a more modest response to output than in model I. The overall picture is that frictions require an increasingly aggressive response of the interest rate to inflation to mitigate shocks that are not dampened by changes in the real wage (model II) and to offset the financial accelerator effect in model III.

We finally seek an improvement on the asymmetric Taylor rule by allowing monetary policy to also respond, data permitting, to the risk premium $\theta_{F,t}$ in the formal sector in model III. The rule now takes the form

$$r_{n,t} \quad = \quad \rho r_{n,t-1} + \theta_\pi \pi_t + \theta_{Fy} y_{F,t} - \theta_{F\theta} \theta_{F,t}\,;\ \theta_\pi,\, \theta_{Fy},\, \theta_{F\theta} > 0 \tag{51}$$

Model	Rule	$[\rho,\ \theta_{\pi F},\ \theta_{yF},\ \theta_{\theta F}]$	Ω_0	σ_r^2	π^*	c_e
III	Asym TR	[0.931, 0.169, 0.000, 0.076]	101.00	0.08	0	0.65

Table 5. Optimized Taylor Rule Responding to the F-sector Risk Premium

The result is shown in Table 5. We now have long-run responses to inflation, output and the risk-premium of 2.45, 0.00 and 1.10 respectively. This more than one-to-one long-run response of the nominal interest rate to changes in the risk-premium brings about a significant welfare improvement in the outcome of $c_e = 0.32\%$. Again with less conservative estimates for the size and persistence of the shocks, this improvement can turn out as considerable.

7 Conclusions

Our main results for the implications of informality for the conduct of monetary policy are as follows. First labour and financial market frictions, the latter assumed to be stronger in the I-sector, cause the time-inconsistency problem to worsen. The importance of commitment therefore increases in economies with a large informal sector with the features we have highlighted. Simple implementable optimized rules that respond only to observed aggregate inflation and F-sector output can be significantly worse in welfare terms than their optimal counterpart, but are still far better than discretion. Simple rules that respond, if possible, to the risk premium in the F-sector result in a significant welfare improvement.

We conclude by discussing a number of caveats and possible directions for future research. Inevitably our results are dependent on our modelling strategy and choice of calibration. Whilst some results not reported suggest that our conclusions are not too sensitive to the latter, there remains the question of alternative models. We have chosen to model labour market frictions as a real wage norm in the Harris-Todaro tradition, but as we discuss in the review of the literature a search-match approach that endogenizes the bargained real wage would pose an interesting alternative. Our model is closed - in the open economy the issue of liability dollarization becomes an important issue giving the finacial accelerator more bite (see Batini *et al.* (2007) and Batini *et al.* (2009)).

Finally it would be desirable to estimate the model by Bayesian methods as is now commonplace in the literature. This leads to the need to properly take into account the lack of observability of this sector in solving for the rational expectations equilibrium and the estimation. This is not done in this paper, nor indeed in the DSGE literature as a whole.[17] This caveat suggests another important direction for research.

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[17]Exceptions are Levine *et al.* (2007) and Levine *et al.* (2009).

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Appendix

A Expressing Summations as Difference Equations

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form

$$\Omega_t = E_t \left[\sum_{k=0}^{\infty} \beta^k X_{t,t+k} Y_{t+k} \right] \tag{A.1}$$

where $X_{t,t+k}$ has the properties $X_{t,t} = 1$ and $X_{t,t+k} = X_{t,t+1} X_{t+1,t+k}$ (for example an inflation, interest or discount rate over the interval $[t, t+k]$).

Lemma

Ω_t can be expressed as

$$\Omega_t = Y_t + \beta E_t \left[X_{t,t+1} \Omega_{t+1} \right] \tag{A.2}$$

Proof

$$\begin{aligned}
\Omega_t &= X_{t,t} Y_t + E_t \left[\sum_{k=1}^{\infty} \beta^k X_{t,t+k} Y_{t+k} \right] \\
&= Y_t + E_t \left[\sum_{k'=0}^{\infty} \beta^{k'+1} X_{t,t+k'+1} Y_{t+k'+1} \right] \\
&= Y_t + \beta E_t \left[\sum_{k'=0}^{\infty} \beta^{k'} X_{t,t+1} X_{t+1,t+k'+1} Y_{t+k'+1} \right] \\
&= Y_t + \beta E_t \left[X_{t,t+1} \Omega_{t+1} \right] \qquad \square
\end{aligned}$$

B The Non-Linear Set-up

For $i = I, F$ the benchmark model without labour market or financial friction is

$$\Lambda_t = \Lambda(C_t, L_{F,t}, L_{I,t}) = \frac{C_t^{(1-\varrho)(1-\sigma)}(n_{F,t}L_{F,t}^{\varrho(1-\sigma)} + (1-n_{F,t})L_{I,t}^{\varrho(1-\sigma)}) - 1}{1-\sigma} \quad \text{(B.1)}$$

$$\text{(B.2)}$$

$$\Lambda_{C,t} = (1-\varrho)C_t^{(1-\varrho)(1-\sigma)-1}(n_{F,t}L_{F,t}^{\varrho(1-\sigma)} + (1-n_{F,t})L_{I,t}^{\varrho(1-\sigma)}) \quad \text{(B.3)}$$

$$\text{(B.4)}$$

$$\Lambda_{L_i,t} = \varrho C_t^{(1-\varrho)(1-\sigma)}(L_{i,t})^{\varrho(1-\sigma)-1} \quad \text{(B.5)}$$

$$\Lambda_{C,t} = \beta E_t\left[(1+R_{t+1})\Lambda_{C,t+1}\right] \quad \text{(B.6)}$$

$$\frac{\Lambda_{L_{i,t}}}{\Lambda_{C,t}} = \frac{W_{i,t}}{P_t} \quad \text{(B.7)}$$

$$L_{i,t} \equiv 1 - h_{i,t} \quad \text{(B.8)}$$

$$Y_{i,t}^W = F(A_{i,t}, n_{i,t}, h_{i,t}, K_{i,t}) = (A_{i,t}n_{i,t}h_{i,t})^{\alpha_i}K_{i,t}^{1-\alpha_i} \quad \text{(B.9)}$$

$$n_{F,t} + n_{I,t} = 1 \quad \text{(B.10)}$$

$$Y_{i,t} = (1-c_i)Y_{i,t}^W \quad \text{(B.11)}$$

$$\frac{P_{i,t}^W}{P_t}F_{h_i,t} = \frac{P_{i,t}^W}{P_t}\frac{\alpha_i Y_{i,t}^W}{h_{i,t}} = \frac{W_{i,t}}{P_t}(1+\tau_{i,t}) \quad \text{(B.12)}$$

$$P_{i,t} = \frac{1}{1-\frac{1}{\zeta_i}}P_{i,t}^W \quad \text{(B.13)}$$

$$K_{t+1} = (1-\delta)K_t + (1-S(Z_t))I_t \quad \text{(B.14)}$$

$$Z_t \equiv \frac{I_t}{I_{t-1}} \quad \text{(B.15)}$$

$$Q_t(1-S(Z_t)-Z_tS'(Z_t)) + E_t\left[\frac{1}{(1+R_{t+1})}Q_{t+1}S'(Z_{t+1})\frac{I_{t+1}^2}{I_t^2}\right] = 1 \quad \text{(B.16)}$$

$$S(Z_t) = \phi_Z(Z_t-(1+g))^2 \quad \text{(B.17)}$$

$$E_t[(1+R_{i,t+1})RP_{i,t+1}] = \frac{E_t\left[(1-\alpha_i)\frac{P_{i,t+1}^W Y_{i,t+1}^W}{P_{t+1}K_{i,t+1}} + (1-\delta)Q_{t+1}\right]}{Q_t} \equiv E_t\left[\frac{X_{i,t+1}}{Q_t}\right] \quad \text{(B.18)}$$

$$Y_{F,t} = C_{F,t} + G_t + I_t \quad \text{(B.19)}$$

$$Y_{I,t} = C_{I,t} \quad \text{(B.20)}$$

$$C_{F,t} = \mathrm{w}\left(\frac{P_{F,t}}{P_t}\right)^{-\mu}C_t \quad \text{(B.21)}$$

$$C_{I,t} = (1-\mathrm{w})\left(\frac{P_{I,t}}{P_t}\right)^{-\mu}C_t \quad \text{(B.22)}$$

$$\frac{P_{F,t}}{P_t} = \frac{1}{\left[\mathrm{w}+(1-\mathrm{w})\mathcal{T}_t^{1-\mu}\right]^{\frac{1}{1-\mu}}} \quad \text{(B.23)}$$

$$\frac{P_{I,t}}{P_t} = \frac{1}{\left[\mathrm{w}\mathcal{T}_t^{\mu-1}+1-\mathrm{w}\right]^{\frac{1}{1-\mu}}} \quad \text{(B.24)}$$

$$\mathcal{T}_t = \mathcal{T}_{t-1} + \Pi_{I,t} - \Pi_{F,t} \quad \text{(B.25)}$$

$$H_{i,t} - \xi_i \beta E_t[\Pi_{i,t+1}^{\zeta_i - 1} H_{i,t+1}] = Y_{i,t}\Lambda_{C,t} \tag{B.26}$$

$$J_{i,t} - \xi_i \beta E_t[\Pi_{i,t+1}^{\zeta_i} J_{i,t+1}] = \left(\frac{1}{1-\frac{1}{\zeta_i}}\right) Y_{i,t}\Lambda_{C,t} MS_{i,t} MC_{i,t} \tag{B.27}$$

$$1 = \xi_i \Pi_{i,t}^{\zeta_i - 1} + (1-\xi_i)\left(\frac{J_{i,t}}{H_{i,t}}\right)^{1-\zeta_i} \tag{B.28}$$

$$MC_{i,t} = \frac{P_{i,t}^W}{P_{i,t}} \tag{B.29}$$

$$\frac{P_{F,t}}{P_t} G_t = n_{F,t} h_{F,t} \tau_{F,t} \frac{W_{F,t}}{P_t} + n_{I,t} h_{I,t} \tau_{I,t} \frac{W_{I,t}}{P_t} \tag{B.30}$$

$$\tau_{I,t} = k\tau_{F,t} \tag{B.31}$$

$$1 + R_t = \frac{1 + R_{n,t-1}}{\Pi_t} \tag{B.32}$$

$$\log\left(\frac{1+R_{n,t}}{1+R_n}\right) = \rho \log\left(\frac{1+R_{n,t-1}}{1+R_n}\right) + \theta \log\left(\frac{\Pi_t}{\Pi}\right) \tag{B.33}$$

$$\log A_{i,t} - \log \bar{A}_{i,t} = \rho_{A_i}(\log A_{i,t-1} - \log \bar{A}_{i,t-1}) + \epsilon_{A_i,t} \tag{B.34}$$

$$\log G_t - \log \bar{G}_t = \rho_G(\log G_{t-1} - \log \bar{G}_{t-1}) + \epsilon_{G,t} \tag{B.35}$$

$$\log MS_{i,t} - \log MS_i = \rho_{MS_i}(\log MS_{i,t-1} - \log MS_i) + \epsilon_{MS_i,t} \tag{B.36}$$

$$\log RPS_{i,t} - \log RPS_i = \rho_{RPS_i}(\log RPS_{i,t-1} - \log RPS_i) + \epsilon_{RPS_i,t} \tag{B.37}$$

In the model without labour or credit market frictions we have

$$\frac{W_{F,t}}{P_t} = \frac{W_{I,t}}{P_t} \tag{B.38}$$

$$R_{F,t} = R_{I,t} = R_t \tag{B.39}$$

It follows from the household FOC that $L_{F,t} = L_{I,t}$ and $h_{F,t} = h_{I,t}$. This completes the model.

With labour and credit market frictions we have[18]

$$\frac{W_{F,t}}{P_t} = RW_t > \frac{W_{I,t}}{P_t} \tag{B.40}$$

$$1 + R_{i,t} \equiv \Theta_{i,t}(1+R_t) \tag{B.41}$$

$$\Theta_{i,t} = s_i\left(\frac{N_{i,t}}{Q_{t-1}K_{i,t}}\right) \; ; \; s'(\cdot) < 0 \tag{B.42}$$

$$N_{i,t+1} = \phi_i V_{i,t} + (1-\phi_i) D_{i,t} \tag{B.43}$$

$$V_{i,t} = (1 + R_{i,t}^k) Q_{t-1} K_{i,t} - \Theta_{i,t}(1+R_t)(Q_{t-1}K_{i,t} - N_{i,t})$$

$$1 + R_{i,t}^k = \frac{X_{i,t}}{Q_{t-1}} \tag{B.44}$$

$$C_{i,t}^e = \frac{1-\phi_i}{\phi_i} N_{i,t} \tag{B.45}$$

$$C_t^{agg} = C_t + C_{I,t}^e + C_{F,t}^e \tag{B.46}$$

[18]The Dynare code for the full model is available from the authors on request.

C Linearization

Define lower case variables $x_t = \log \frac{X_t}{\bar{X}_t}$ if X_t has a long-run trend or $x_t = \log \frac{X_t}{X}$ otherwise where X is the steady state value of a non-trended variable. For variables $n_{F,t}$, $n_{I,t}$ and h_t define $\hat{x}_t = \log \frac{x_t}{x}$; $r_{n,t} \equiv \log\left(\frac{1+R_{n,t}}{1+R_n}\right)$; $\pi_{i,t} \equiv \log\left(\frac{1+\Pi_{i,t}}{1+\Pi_i}\right)$, $i = I, F$ are log-linear *gross* interest and inflation rates.

First consider the model **without financial frictions**. Our linearized model about the BGP zero-inflation steady state then takes the state-space form form

$$
\begin{aligned}
a_{F,t+1} &= \rho_{aF} a_{F,t} + \varepsilon_{aF,t+1} && \text{(C.1)}\\
a_{I,t+1} &= \rho_{aI} a_{I,t} + \varepsilon_{aI,t+1} && \text{(C.2)}\\
g_{t+1} &= \rho_g g_t + \varepsilon_{g,t+1} && \text{(C.3)}\\
u_{F,t+1} &= \rho_{uF} u_{F,t} + \varepsilon_{uF,t+1} && \text{(C.4)}\\
u_{I,t+1} &= \rho_{uI} u_{I,t} + \varepsilon_{uI,t+1} && \text{(C.5)}\\
rps_{t+1} &= \rho_{rps} rps_t + \varepsilon_{rps,t+1} && \text{(C.6)}\\
\tau_t &= \tau_{t-1} + \pi_{I,t} - \pi_{F,t} && \text{(C.7)}\\
k_t &= \frac{1-\delta}{1+g} k_{t-1} + \frac{\delta+g}{1+g} i_t && \text{(C.8)}\\
E_t[\lambda_{C,t+1}] &= \lambda_{C,t} - E_t[r_{t+1}] && \text{(C.9)}\\
\beta E_t[\pi_{F,t+1}] &= \pi_{F,t} - \lambda_F(mc_{F,t} + u_{F,t}) && \text{(C.10)}\\
\beta E_t[\pi_{I,t+1}] &= \pi_{I,t} - \lambda_I(mc_{I,t} + u_{I,t}) && \text{(C.11)}\\
x_{F,t} : E_t[r_{F,t+1}] &= E_t[x_{F,t+1}] - q_t && \text{(C.12)}\\
& && \text{(C.13)}
\end{aligned}
$$

$$
\begin{aligned}
i_t : \quad \left(1 + \frac{1+g}{1+R}\right) i_t &= \frac{1+g}{1+R} E_t i_{t+1} + i_{t-1} + \frac{1}{(1+g)^2 S''(1+g)} q_t && \text{(C.14)}\\
r_{F,t} &= r_{I,t} = r_t && \text{(C.15)}
\end{aligned}
$$

Outputs are defined by

$$
\begin{aligned}
r_t &= r_{n,t-1} - \pi_t && \text{(C.16)}\\
\pi_t &= \mathrm{w}\pi_{F,t} + (1-\mathrm{w})\pi_{I,t} && \text{(C.17)}\\
c_t : \ \lambda_{C,t} &= -(1 + (\sigma-1)(1-\varrho))c_t \\
&+ \frac{n_F(L_F^{\varrho(1-\sigma)} - L_I^{\varrho(1-\sigma)})\hat{n}_{F,t} + \varrho(1-\sigma)(n_F L_F^{\varrho(1-\sigma)} \ell_{F,t} + (1-n_F) L_I^{\varrho(1-\sigma)} \ell_{I,t})}{n_F L_F^{\varrho(1-\sigma)} + (1-n_F) L_I^{\varrho(1-\sigma)}} \\
&= -(1 + (\sigma-1)(1-\varrho))c_t + \varrho(1-\sigma)(n_F \ell_{F,t} + (1-n_F)\ell_{I,t}) \text{ if } L_{I,t} = L_{F,t} && \text{(C.18)}\\
\ell_{F,t} &= -\frac{h_F}{1-h_F} \hat{h}_{F,t} && \text{(C.19)}\\
\ell_{I,t} &= -\frac{h_I}{1-h_I} \hat{h}_{I,t} && \text{(C.20)}
\end{aligned}
$$

$$
\begin{aligned}
u_{L_I,t} &= -(\sigma-1)(1-\varrho)c_t + (1+\varrho(\sigma-1))\frac{h_I}{1-h_I}\hat{h}_{I,t} && \text{(C.21)}\\
u_{L_F,t} &= -(\sigma-1)(1-\varrho)c_t + (1+\varrho(\sigma-1))\frac{h_F}{1-h_F}\hat{h}_{F,t} && \text{(C.22)}\\
w_{I,t}-p_t &= u_{L_I,t}-\lambda_{C,t} && \text{(C.23)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{h}_{F,t}:\ w_{F,t}-p_t &= u_{L_F,t}-\lambda_{C,t} && \text{(C.24)}\\
w_{F,t}-p_t &= \omega(w_{I,t}-p_t) && \text{(C.25)}\\
c_{F,t} &= c_t+\mu(1-\mathrm{w})\tau_t && \text{(C.26)}\\
c_{I,t} &= c_t-\mu\mathrm{w}\tau_t && \text{(C.27)}\\
\hat{n}_{F,t}:\ y_{F,t} &= \alpha_F(a_{F,t}+\hat{n}_{F,t}+\hat{h}_{F,t})+(1-\alpha_F)k_{F,t} && \text{(C.28)}\\
\hat{h}_{I,t}:\ y_{I,t} &= \alpha_I(a_{I,t}+\hat{n}_{I,t}+\hat{h}_{I,t})+(1-\alpha_I)k_{I,t} && \text{(C.29)}\\
\hat{n}_{I,t} &= -\frac{n_F}{n_I}\hat{n}_{F,t} && \text{(C.30)}\\
mc_{F,t} &= w_{F,t}-p_t+\frac{\tau_F}{1+\tau_F}\hat{\tau}_{F,t}+(1-\mathrm{w})\tau_t-\alpha_F a_{F,t}\\
&+ (1-\alpha_F)(\hat{n}_{F,t}+\hat{h}_{F,t}-k_{F,t}) && \text{(C.31)}\\
mc_{I,t} &= w_{I,t}-p_t+\frac{\tau_I}{1+\tau_I}\hat{\tau}_{I,t}-\mathrm{w}\tau_t-\alpha_I a_{I,t}\\
&+ (1-\alpha_I)(\hat{n}_{I,t}+\hat{h}_{I,t}-k_{I,t}) && \text{(C.32)}\\
y_{I,t} &= y^W_{I,t}=c_{I,t} && \text{(C.33)}\\
y_{F,t} &= y^W_{F,t}=c_{yF}c_{F,t}+i_{yF}i_t+g_{yF}\,g_t && \text{(C.34)}\\
g_t &= -(1-\mathrm{w})\tau_t+\frac{n_F\tau_F}{n_F\tau_t+n_I\tau_I}(\hat{n}_{F,t}+\hat{h}_{F,t}+\hat{\tau}_{F,t}+w_{F,t}-p_t)\\
&+ \frac{n_I\tau_I}{n_F\tau_t+n_I\tau_I}(\hat{n}_{I,t}+\hat{h}_{I,t}+\hat{\tau}_{I,t}+w_{I,t}-p_t)\\
\hat{\tau}_{I,t} &= \hat{\tau}_{F,t} && \text{(C.35)}
\end{aligned}
$$

$$
\begin{aligned}
k_{F,t}:\ k_t &= \frac{\bar{K}_{F,t}}{\bar{K}_t}k_{F,t}+\frac{\bar{K}_{I,t}}{\bar{K}_t}k_{I,t} && \text{(C.36)}\\
q_t:\ x_{F,t} &= (R_F+\delta)[y^W_{F,t}-k_{F,t}+mc_{F,t}-(1-w)\tau_t]+(1-\delta)q_t && \text{(C.37)}\\
k_{I,t}:\ x_{I,t} &= (R_I+\delta)[y^W_{I,t}-k_{I,t}+mc_{I,t}+w\tau_t]+(1-\delta)q_t && \text{(C.38)}\\
x_{I,t}:r_{I,t} &= x_{I,t}-q_{t-1}\\
\text{replaces } E_t[r_{I,t+1}] &= E_t[x_{I,t+1}]-q_t && \text{(C.39)}\\
r_{F,t} &= r_{I,t}=r_t+rps_t && \text{(C.40)}
\end{aligned}
$$

where $\lambda_i \equiv \frac{(1-\beta\xi_i)(1-\xi_i)}{\xi_i}$, and $\tau_i \equiv \frac{\bar{\tau}_i}{W_t/P}$ $i=I,F$. Note that (C.18) defines c_t, (C.28) defines $\hat{n}_{F,t}$ and (C.29) defines $\hat{h}_t$. Let $\tau_I=(1-k)\tau_F$ where $k\in[0,1]$ to allow taxation to be enforced in the informal sector. Also (C.26) and (C.27) implies $c_t=\mathrm{w}c_{F,t}+(1-\mathrm{w})c_{I,t}$

With financial frictions we add the equations for $i = I, F$

$$
\begin{aligned}
n_{i,t+1} &= \frac{\xi_{ei}}{1+g}\Theta_i(1+R)\left[\frac{1}{n_{ki}}r^k_{i,t} + n_{i,t} + \left(1 - \frac{1}{n_{ki}}\right) r_{i,t}\right] && \text{(C.41)}\\
r^k_{i,t} &= x_{i,t} - q_{t-1} && \text{(C.42)}\\
r_{i,t} &= \theta_{i,t} + r_t && \text{(C.43)}\\
\theta_{i,t} &= \chi_i(n_{i,t} - k_{i,t} - q_{t-1}) + \epsilon_{P_i,t} && \text{(C.44)}\\
c^e_{i,t} &= n_{i,t} && \text{(C.45)}\\
c^a_t &= \frac{C}{C^a}c_t + \frac{C^e_I}{C^a}c^e_{I,t} + \frac{C^e_F}{C^a}c^e_{F,t} && \text{(C.46)}\\
c_{F,t} &= c^a_t + \mu(1-\mathrm{w})\tau_t && \text{(C.47)}\\
c_{I,t} &= c^a_t - \mu \mathrm{w}\tau_t && \text{(C.48)}
\end{aligned}
$$

D The Hamiltonian Quadratic Approximation of Welfare

Consider the following general deterministic optimization problem

$$
\max \sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) \;\; s.t. \;\; X_t = f(X_{t-1}, W_t) \qquad \text{(D.1)}
$$

where X_{t-1} is vector of state variables and W_{t-1} a vector of instruments.[19] There are given initial and the usual tranversality conditions. For our purposes, we consider this as including models with forward-looking expectations, so that the optimal solution to the latter setup is the pre-commitment solution. Suppose the solution converges to a steady state X, W as $t \to \infty$ for the states X_t and the policies W_t. Define $x_t = X_t - X$ and $w_t = W_t - W$ as representing the first-order approximation to absolute deviations of states and policies from their steady states.[20]

The Lagrangian for the problem is defined as,

$$
\sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda_t^T (X_t - f(X_{t-1}, W_t))] \qquad \text{(D.2)}
$$

[19] An alternative representation of the problem is $U(X_t, W_t)$ and $E_t[X_{t+1}] = f(X_t, W_t)$ where X_t includes forward-looking non-predetermined variables and $E_t[X_{t+1}] = X_{t+1}$ for the deterministic problem where perfect foresight applies. Whichever one uses, it is easy to switch from one to the other by a simple re-definition. Note that Magill (1977) adopted a continuous-time model without forward-looking variables. As we demonstrate in Levine *et al.* (2008b), although the inclusion of forward-looking variables significantly alters the nature of the optimization problem, these changes only affect the boundary conditions and the second-order conditions, but not the steady state of the optimum which is all we require for LQ approximation.

[20] Alternatively $x_t = (X_t - X)/X$ and $w_t = (W_t - W)/W$, depending on the nature of the economic variable. Then the Theorem follows in a similar way with an appropriate adjustment to the Jacobian Matrix.

so that a necessary condition for the solution to (D.1) is that the Lagrangian is stationary at all $\{X_s\}$, $\{W_s\}$ i.e.

$$U_W + \lambda_t^T f_W = 0 \qquad U_X - \frac{1}{\beta}\lambda_{t-1}^T + \lambda_t^T f_X = 0 \tag{D.3}$$

Assume a steady state λ for the Lagrange multipliers exists as well. Now define the Hamiltonian $H_t = U(X_{t-1}, W_t) + \lambda^T f(X_{t-1}, W_t)$. The following is the discrete time version of Magill (1977):

Theorem: If a steady state solution (X, W, λ) to the optimization problem (D.1) exists, then any perturbation (x_t, w_t) about this steady state can be expressed as the solution to

$$\max \frac{1}{2}\sum_{t=0}^{\infty} \beta^t \left[\begin{array}{cc} x_{t-1} & w_t \end{array} \right] \left[\begin{array}{cc} H_{XX} & H_{XW} \\ H_{WX} & H_{WW} \end{array} \right] \left[\begin{array}{c} x_{t-1} \\ w_t \end{array} \right] \quad s.t.\ \ x_t = f_X x_{t-1} + f_W w_t \tag{D.4}$$

where H_{XX}, etc denote second-order derivatives evaluated at (X, W). This can be directly extended to the case incorporating disturbances.

Thus our general procedure is as follows:

1. Set out the deterministic non-linear problem for the Ramsey Problem, to maximize the representative agents' utility subject to non-linear dynamic constraints.

2. Write down the Lagrangian for the problem.

3. Calculate the first order conditions. We do not require the initial conditions for an optimum since we ultimately only need the steady-state of the Ramsey problem.

4. Calculate the steady state of the first-order conditions. The terminal condition implied by this procedure is such that the system converges to this steady state.

5. Calculate a second-order Taylor series approximation, about the steady state, of the Hamiltonian associated with the Lagrangian in 2.

6. Calculate a first-order Taylor series approximation, about the steady state, of the first-order conditions and the original constraints.

7. Use 4. to eliminate the steady-state Lagrangian multipliers in 5. By appropriate elimination both the Hamiltonian and the constraints can be expressed in minimal form. This then gives us the accurate LQ approximation of the original non-linear optimization problem in the form of a minimal linear state-space representation of the constraints and a quadratic form of the utility expressed in terms of the states.